Attachment No 3 to the Application for Entering the Habilitation Procedure
-summary of professional accomplishments

1. Name: Ewa Jolanta Rak
2. Academic degrees:

   2.1 Master in Mathematics: University of Rzeszów, Faculty of Mathematics and Natural Sciences, July 3, 2003.
   Thesis title: *Modifications of Łukasiewicz operations* (in Polish).

   2.2 PhD in Mathematics: Pedagogical University of Cracow, Faculty of Mathematics, Physics and Technical Sciences, July 1, 2009.
   Thesis title: *Distributivity of increasing operations* (in Polish).

3. Employments in scientific institutions:

   3.1 October 1, 2003 - February 28, 2011: Research Assistant, University of Rzeszów, Faculty of Mathematics and Natural Sciences.

   3.2 March 1, 2011 - now: Assistant Professor, University of Rzeszów, Faculty of Mathematics and Natural Sciences.

4. Achievement resulting from Article 16 Paragraph 2 of the Act of 14 March 2003 on Academic Degrees and Title and on Degrees and Title in the Field of Art is a set of publications entitled:

   4.1 **Solutions of the distributivity and modularity equations for some classes of aggregation functions**

   4.2 **List of publications included in the scientific achievement:**

[R1] J. Drewniak, **E. Rak**, *Distributivity inequalities of monotonic operations*, Fuzzy Sets and Systems 191 (2012), 62-71.

[R2] E. Rak, *The distributivity property of increasing binary operations*, Fuzzy Sets and Systems 232 (2013), 110-119.

[R3] P. Drygaś, **E. Rak**, *Distributivity equation in the class of semi-t-operators*, Fuzzy Sets and Systems 291 (2016), 66-81.

[R4] P. Drygaś, **E. Rak**, *Distributivity equation in the class of 2-uninorms*, Fuzzy Sets and Systems 291 (2016), 82-97.

[R5] P. Drygaś, F. Qin, **E. Rak**, *Left and right distributivity equations for semi-t-operators and uninorms*, Fuzzy Sets and Systems 325 (2017), 21-34.

[R6] W. Fechner, **E. Rak**, L. Zedam, *The modularity law of some classes of aggregation operators*, Fuzzy Sets and Systems (2017) http://dx.doi.org/10.1016/j.fss.2017.03.010.

4.3 **The description of the scientific goal of the foregoing papers and results obtained, jointly with their potential applications.**

The aim of the set of publications is to examine some problems of functional equations in two variables (more precisely distributivity and modularity equations) for functions covered by a relatively new and distinct direction of research - aggregation theory. Such a combination gives the possibility of using obtained results, both in other mathematical disciplines and in various applied areas. Tools and the proof techniques applied are in-depth (significantly improved) versions of the standard methods used for solving similar problems, mainly due to the use of the minimal set of assumptions as the most desirable in practical applications.

Introduction

Distributivity of multiplication with respect to the addition occurs naturally in the arithmetic of real numbers, in vectors and matrices calculus. A derivative and a definite integral are distributive with respect to the addition, as well. The distributivity axiom occurs in the definitions of fields and rings. In general, it specifies the relationship between two binary operations.

Definition 1. [cf. [1], p. 318] Let F and G be some binary operations in the non-empty set X. We say that F is distributive over G if for all $x, y, z \in X$ the following equalities are fulfilled:

$$F(x, G(y, z)) = G(F(x, y), F(x, z)) \qquad (LD)$$

$$F(G(y, z), x) = G(F(y, x), F(z, x)) \qquad (RD)$$

We can talk about the left distributivity of F with respect to G when only the first (LD) of the above conditions is satisfied, or about the right distributivity when only the second condition (RD) is fulfilled. We should point out that a commutative operation F distributive on one-side is distributive on both sides.

A more general approach is to treat the distributivity axiom as a functional equation with one or two unknown functions (operations).

Solutions of distributivity equation largely depend on the choice of the class of functions, in which we are seeking solutions. Primarily, these studies included the auto-distributivity equation (when $F = G$) in the class of continuous reducible (which implied the strict monotonicity) and symmetric functions defined in the real interval, see for example, M. Hosszú [33], whose solutions were characterized by quasilinear weighted means. Next papers dealt with a one-sided distributivity equation for functions under the strict monotonicity and twice differentiability assumptions instead of continuity, which indicated that the left distributivity was substantially independent of the right distributivity (see e.g., M. Hosszú [34]). Significant considerations on the problem of distributivity (up to 1965) are included in J. Aczél's monograph [1], who also made the effort to characterize solutions of such equations. In particular, he pointed out solutions of the right distributivity equation for functions bounded from below with respect to continuous, increasing and associative functions with a both-sided neutral element (see Chapter 7.1.3, Theorem 6). From subsequent publications noteworthy are papers of A. Lundberg [41] and [42], devoted to the generalized distributivity equation in the class of continuous functions. The above results are subsequently used in probability theory, partial differential equations theory and vector and matrix equations theory, as J. Aczél emphasized in [1], see among others pages 321, 342 and 372.

At present, many studies are dealing with the distributivity equation for different operations defined on the unit interval that are essential in decision making and utility theories [24, 36, 43], fuzzy logic theory, integration theory [57] or in image processing [30, 51].

Due to the demand for these practical applications, discussion on the distributivity equation between various functions, including aggregation functions, have revived (see e.g., C. Alsina et al. [5], M. Carbonell et al. [16], J. Dombi [19], D. Dubois and H. Prade [23], J. Drewniak ([21], pp. 51 and 89-90, [20]). T. Calvo [13] characterized, among others, solutions of the distributivity equation for averaging and quasilinear functions. In the paper [59] there has been published an open problem involving the distributivity between two special classes of aggregation functions i.e. uninorms and continuous triangular conorms.

Among many publications dealing with the problem of distributivity in the unit interval there are results for triangular norms and conorms in papers of C. Alsina [3] and C. Bertoluzza, V. Doldi [12] and results for uninorms and nullnorms in papers of M. Mas et al. ([46], [47]) and D. Ruiz, J. Torrens ([53], [55]) as well. Some results are also related to the distributivity of fuzzy implications, as the paper of M. Baczyński [9] or implications over uninorms of D. Ruiz and J. Torrens ([54], [56]).

In turn the modular axiom (formerly modular law) is defined as follows.

Definition 2 ([45])**.** Let $F, G : [0,1]^2 \to [0,1]$. We say that F is modular over G if for all $x, y, z \in [0,1]$ the following condition holds

$$z \leqslant x \Rightarrow F(x, G(y,z)) = G(F(x,y), z). \tag{1}$$

The condition (1) can also be perceived as a generalized restricted associativity equation (automodularity for $F = G$) as well as the weakened distributivity equation (for example, lattices of semigroups and other algebraic structures are modular but not distributive), which led to their more detailed examination.

In recent years, more and more attention has been devoted to general approach treating the modularity axiom as a functional equation with one or two unknown functions (operations). Among publications on a modularity equation for aggregation functions the most important are papers of M. Carbonell et al. [16], Q. Feng [26], M. Mas et al. [45] and H. Zhan et al. [67].

The search for solutions to problems of distributivity and modularity is in fact not a simple matter, especially when we seek to minimize the set of assumptions of considered functions. In the case of less popular modularity equation I was also aware that considering it often leads to a kind of failure i.e. the lack of solutions. However, a comparison of solutions for both equations was the goal of the research undertaken, and thus, papers (in chronological order) [R4], [R10], [R6] have just been created.

Now, we will consider the aggregation functions in the interval [0,1], limited to the binary case.

Definition 3 (cf. [31], Chapter 1)**.** A binary aggregation function is a mapping $A : [0,1]^2 \to [0,1]$ such that
$A1)A(0,0) = 0$ and $A(1,1) = 1$ (boundary conditions);
$A2)A$ is increasing in both variables i.e. $A(x,y) \leqslant A(z,t)$ if $(x,y) \leqslant (z,t)$.
An aggregation function is called a mean if it is idempotent i.e. $A(x,x) = x$ in $[0,1]$.

Aggregation functions are a useful generalization of means. Recently, there have been several monographs on the theory and applications of aggregation functions: *Aggregation Functions (Encyclopedia of Mathematics and Its Applications)* [31] (M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap), *Aggregation Operators: New Trends and Applications* [15] (T. Calvo, G. Mayor i R. Mesiar) and *Aggregation Functions: A Guide for Practitioners* [10] (G. Beliakov, A. Pradera, T. Calvo). Triangular norms and conorms are described in *Triangular Norms* [38] (E.P. Klement, R. Mesiar, E. Pap) and *Associative Functions: Triangular Norms and Copulas* [6] (C. Alsina, M.J. Frank, B. Schweizer), and means in - *A Practical Guide to Averaging Functions* [11] (G. Beliakov, H. Bustince, T. Calvo).

In the area of interest of this theory there is a systematic study of the properties of these functions, their relationships and new construction methods adapted to specific practical applications, mainly in mathematical and computational statistics, computational geometry, in data analysis, decision support systems, recognition and image processing, artificial intelligence, databases, fuzzy control or economics.

Given the diversity of aggregation functions (operations) they are grouped into different classes such as means, triangular norms and conorms, copulas, Choquet and Sugeno integrals, uninorms and nullnorms. However, we will focus on those that are important in further considerations.

Definition 4 (see [R10])**.** Let $e \in [0,1]$. By $\mathcal{N}_e$ we denote the family of all operations $F : [0,1]^2 \to [0,1]$ which are increasing with respect to both variables and have a neutral element $e \in [0,1]$.

Definition 5 ([65])**.** Let $e \in [0,1]$. An operation $F \in \mathcal{N}_e$ which is additionally associative and commutative is called a uninorm with a neutral element e. The family of all uninorms is denoted by $\mathcal{U}_e$.

The general structure of this operation is as follows.

Theorem 1 ([R10])**.** *Let $e \in (0,1)$. $F \in \mathcal{N}_e$ if and only if*

$$F(x,y) = \begin{cases} eA\left(\frac{x}{e}, \frac{y}{z}\right) & \text{if } (x,y) \in [0,e]^2 \\ k + (1-e)B\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2 \\ C(x,y) & \text{if } (x,y) \in D_e \end{cases}, \tag{2}$$

where $D_e = [0,e) \times (e,1] \cup (e,1] \times [0,e)$, $A : [0,e]^2 \to [0,e]$ is increasing with a neutral element e $B : [e,1]^2 \to [e,1]$ is increasing with a neutral element e and $C : D_e \to [0,1]$ is an increasing function fulfilling inequalities $\min(x,y) \leqslant C(x,y) \leqslant \max(x,y)$ *for* $(x,y) \in D_e$.

Definition 6 ([R10])**.** Let $e \in [0,1]$. By $\mathcal{N}_e^{\max}$ ($\mathcal{N}_e^{\min}$) we denote the family of all operations $F \in \mathcal{N}_e$ fulfilling the additional condition:

$$F(0,x) = F(x,0) = x \text{ for all } x \in (e,1] \quad (F(1,x) = F(x,1) = x \text{ for all } x \in [0,e)).$$

Moreover,

$$\mathcal{N}^{\max} = \bigcup_e \mathcal{N}_e^{\max}, \quad \mathcal{N}^{\min} = \bigcup_e \mathcal{N}_e^{\min}.$$

Definition 7 ([R10])**.** Let $k \in [0,1]$. By $\mathcal{Z}_k$ ($\mathcal{Z}_s$ in [R1]) we denote the family of all increasing operations $G : [0,1]^2 \to [0,1]$ having neutral elements $e = 0$ on $[0,k]$ and $e = 1$ on $[k,1]$. The notation $\mathcal{Z}_k$ means that k is a zero element of G (its existence follows from the monotonicity and and neutral elements).

Definition 8 ([14])**.** Let $k \in [0,1]$. An operation $G \in \mathcal{Z}_k$ which is additionally associative and commutative is called a nullnorm with a zero element k. The family of all nullnorms is denoted by $\mathcal{V}$.

Remark 1. In particular, $\mathcal{N}_0^{\min} = \mathcal{N}_0^{\max} = \mathcal{Z}_0 = \mathcal{N}_1$ and $\mathcal{N}_1^{\min} = \mathcal{N}_1^{\max} = \mathcal{Z}_1 = \mathcal{N}_0$, where $\mathcal{N}_1$ ($\mathcal{N}_0$) includes increasing operations with a neutral element $e = 1$ ($e = 0$).
An associative and commutative operation from the class $\mathcal{N}_1$ ($\mathcal{N}_0$) is called a triangular norm (triangular conorm) (t-norm (t-conorm) for short) and is denoted by T (S) (see [38], pp. 6 and 13). Triangular norms and conorms are ordered commutative semigroups in $[0,1]$ with a neutral element at the end of the unit interval.

The general structure of an operation from the family of $\mathcal{Z}_k$ presents the following theorem.

Theorem 2 ([R10])**.** *Let $k \in (0,1)$, $G : [0,1]^2 \to [0,1]$. $G \in \mathcal{Z}_k$ if and only if*

$$G(x,y) = \begin{cases} kA\left(\frac{x}{k}, \frac{y}{z}\right) & \text{if } (x,y) \in [0,k]^2 \\ k + (1-k)B\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } (x,y) \in [k,1]^2 \\ k & \text{if } (x,y) \in D_k \end{cases},$$

where $A : [0,k]^2 \to [0,k]$ is increasing with a neutral element $e = 0$ and $B : [k,1]^2 \to [k,1]$ is increasing with a neutral element $e = 1$.

Taking into account the fact that the unique idempotent operations from the classes $\mathcal{N}_1$ and $\mathcal{N}_0$ are min and max, respectively we immediately obtain two unique idempotent uninorms from families $\mathcal{N}_e^{\min}$ and $\mathcal{N}_e^{\max}$, denoted by $U^{\min}$ and $U^{\max}$, and a unique idempotent nullnorm denoted by V_k (see Fig. 1 and 2).

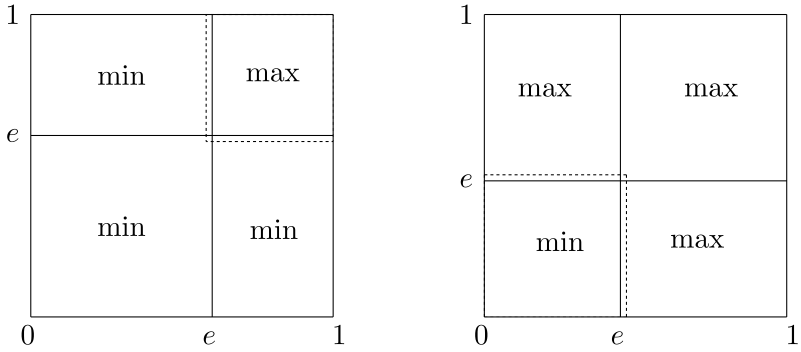

Figure 1: The structure of idempotent uninorms from classes $\mathcal{N}_e^{\min}$ and $\mathcal{N}_e^{\max}$.

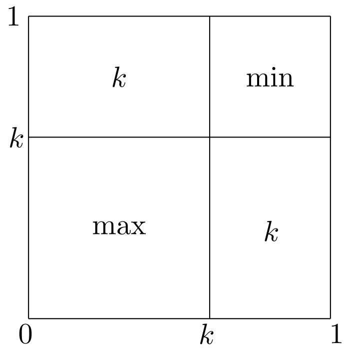

Figure 2: The structure of an idempotent nullnorm V_k.

Uninorms $\mathcal{U}_e \subset \mathcal{N}_e$ and nullorms (equivalently t-operators [44]) $\mathcal{V} \subset \mathcal{Z}_k$, as mixed (also known as compensatory) classes of aggregation operations, are interesting because their structures are a special combination of triangular norms and triangular conorms, and thus have been proved to be useful in many fields like fuzzy logic, expert systems, neural networks, utility theory and fuzzy system modeling (see e.g., [24], [28], [39], [43], [64]).

Due to this quantity of applications, extensive theoretical studies have been undertaken covering the characterization of solutions of functional equations for aggregations, including mainly a distributivity equation. The lack of distributivity is a big problem in any algebraic transformations, and therefore also in computer modeling (see e.g., [17]). In general, aggregations are not distributive from each other, and still less mutually distributive. In my opinion the best is to illustrate this problem on the example of means (Table 1), t-norms and t-conorms (Table 2). As a result, between the right and the left side of the distributivity equation (LD) there may occur four different relations $L \leqslant P$, $L \geqslant P$, $L = P$ and $L \parallel P$ (both sides of (LD) are incomparable), as was summarized in Table 3.

Table 1: Examples of basic means in [0,1] (see [R11]).

Mean	Mean	Name
$M_\wedge(x,y) = \min(x,y)$	$M_\vee(x,y) = \max(x,y)$	Minimum and maksimum
$M_A(x,y) = \frac{x+y}{2}$	$M_G(x,y) = \sqrt{xy}$	Aritmetic and geometric means
$M_H(x,y) = \begin{cases} 0, & x = y = 0 \\ \frac{2xy}{x+y}, & \textit{poza tym} \end{cases}$	$M_P(x,y) = \sqrt{\frac{x^2+y^2}{2}}$	Harmonic and power means
$P_1(x,y) = x$	$P_2(x,y) = y$	The left-hand and the right-hand projections
$M_\lambda(x,y) = \lambda x + (1-\lambda)y, \;\; \lambda \in [0,1]$		Linear means

Table 2: Examples of basic triangular norms and conorms (see [R11]).

T-norm	T-conorm	Name
$T_M(x,y) = \min(x,y)$	$S_M(x,y) = \max(x,y)$	Lattice operations
$T_P(x,y) = x \cdot y$	$S_P(x,y) = x + y - x \cdot y$	Algebraic operations
$T_L(x,y) = \max(x+y-1, 0)$	$S_L(x,y) = \min(x+y, 1)$	Łukasiewicz operations
$T_D(x,y) = \begin{cases} \min(x,y), & \max(x,y) = 1 \\ 0, & \max(x,y) < 1 \end{cases}$	$S_D(x,y) = \begin{cases} \max(x,y), & \min(x,y) = 0 \\ 1, & \min(x,y) > 0 \end{cases}$	Drastic operations

Table 3: Distributivity of t-norms, t-conorms and means given by Tab. 1 and Tab. 2 (see [R11]).

$\mathbf{F \setminus G}$	$\mathbf{S_D}$	$\mathbf{S_L}$	$\mathbf{S_P}$	$\mathbf{S_M}$	$\mathbf{M_P}$	$\mathbf{M_A}$	$\mathbf{M_G}$	$\mathbf{M_H}$	$\mathbf{T_M}$	$\mathbf{T_P}$	$\mathbf{T_L}$	$\mathbf{T_D}$
$\mathbf{S_D}$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$\geqslant$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$\|$	$\|$
$\mathbf{S_L}$	$\leqslant$	$\leqslant$	$\|$	$=$	$\geqslant$	$\geqslant$	$\|$	$\|$	$=$	$\|$	$\|$	$\|$
$\mathbf{S_P}$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$=$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$\geqslant$	$\geqslant$
$\mathbf{S_M}$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$\leqslant$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$\geqslant$	$\geqslant$
$\mathbf{M_P}$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$=$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$\geqslant$	$\geqslant$
$\mathbf{M_A}$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$=$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$\geqslant$	$\geqslant$
$\mathbf{M_G}$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$\geqslant$	$=$	$\leqslant$	$=$	$\geqslant$	$\|$	$\geqslant$
$\mathbf{M_H}$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$\geqslant$	$\geqslant$	$=$	$=$	$\|$	$\|$	$\geqslant$
$\mathbf{T_M}$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$\geqslant$	$\geqslant$	$\geqslant$	$\geqslant$	$=$	$\geqslant$	$\geqslant$	$\geqslant$
$\mathbf{T_P}$	$\leqslant$	$\leqslant$	$\leqslant$	$=$	$=$	$=$	$=$	$=$	$=$	$\geqslant$	$\geqslant$	$\geqslant$
$\mathbf{T_L}$	$\|$	$\|$	$\|$	$=$	$\leqslant$	$\leqslant$	$\geqslant$	$\geqslant$	$=$	$\|$	$\geqslant$	$\geqslant$
$\mathbf{T_D}$	$\|$	$\|$	$\leqslant$	$=$	$\leqslant$	$\leqslant$	$=$	$=$	$=$	$\geqslant$	$\geqslant$	$\geqslant$

Despite the difficulties mentioned above, I opted for setting out those pairs of aggregations of parametric families that satisfy the distributivity equation (modularity equation). In the set of papers [R1] - [R6] this goal has been reached. In fact, I focused on solving the problem of distributivity and modularity for different classes of aggregation functions. The research in [R1] - [R2] was motivated by several unsolved issues included in the summary of my doctoral dissertation. Generalizations of results to new classes of aggregation functions required some improvements of previously used tools and the proof techniques.
I will present the overview of the most important results included in the scientific achievement applying the chronological order to undertaken research problems.

The problem of subdistributivity or superdistributivity in the case of the lack of distributivity

The main results of [R1] (with co-author J. Drewniak) are mainly devoted to pairs of non-distributive weak algebraic operations from the families $\mathcal{Z}_k$ and $\mathcal{N}_e$, more precisely, indicating the conditions guaranteeing subdistributivity or superdistributivity of these pairs of operations in the case of the significant lack of their distributivity. These results simultaneously complemented the results of [R7] - [R10]. In particular, we described families of increasing binary operations subdistributive or superdistributive with respect to idempotent uninorms and nullnorms.

If in Definition 1 an equality is replaced by inequalities " $\leqslant$ " or " $\geqslant$ ", respectively and $X = [0,1]$, then for $x, y, z \in [0,1]$ we say that
F is left (right) subdistributive with respect to G if

$$F(x, G(y,z)) \leqslant G(F(x,y), F(x,z)) \quad (F(G(y,z),x) \leqslant G(F(y,x), F(z,x))), \tag{3}$$

F is left (right) superdistributive with respect to G if

$$F(x, G(y,z)) \geqslant G(F(x,y), F(x,z)) \quad (F(G(y,z),x) \geqslant G(F(y,x), F(z,x))). \tag{4}$$

The motivation for such consideration was the paper of C. Alisna [4] and negative results observed by T. Calvo [13] stating, that pairs of triangular norms and conorms are not need to be distributive. Moreover, the paper of M. Mas et al. [46] brought additional information about distributivity and non-distributivity for pairs of uninorms and nullnorms. We established that this lack of distributivity can be replaced by distributivity inequalities as for example, in the lattice theory dealing with the subdistributivity and the superdistributivity.

Characterizations of distributive F and G for $F, G \in \mathcal{Z}_k \cup \mathcal{N}_e^{\min} \cup \mathcal{N}_e^{\max}$ were made in papers [R7] - [R10]. Distributivity condition (LD) or (RD) implies the idempotency of operation G. All positive (+) and negative (-) results (denoted by Res.) obtained in these papers for the left distributivity equation are summarized in the table below,

F/G	V_k			$U_f^{\min}$			$U_f^{\max}$		
	Case	Res.	Reference	Case	Res.	Reference	Case	Res.	Reference
$\mathcal{Z}_s$	$s \leqslant k$	+	[R8], Th. 4	$s \leqslant f$	+	[R11], Th. 18	$f \leqslant s$	+	[R11], Th. 16
	$k < s$	+	[R8], Th. 5	$f < s$	-	[R11], Th. 18	$s < f$	-	[R11], Th. 16
$\mathcal{N}_e^{\min}$	$k < e$	+	[R11], Th. 10	$f \leqslant e$	+	[R9], Th. 3	$0 = f < e$	+	[R10], Th. 3
	$e \leqslant k = 1$	+	[R11], Th. 14	$e < f = 1$	+	[R9], Th. 1	$0 < f < e$	-	[R10], Th. 3
	$e \leqslant k < 1$	-	[R11], Th. 14	$e < f < 1$	-	[R9], Th. 1	$0 < f < e < 1$	$\Vert$	[R10], Th. 5
$\mathcal{N}_e^{\max}$	$e < k$	+	[R11], Th. 9	$e < f = 1$	+	[R10], Th. 4	$e \leqslant f$	+	[R9], Th. 4
	$0 = k \leqslant e$	+	[R11], Th. 13	$e < f < 1$	-	[R10], Th. 4	$0 = f < e$	+	[R9], Th. 2
	$0 < k \leqslant e$	-	[R11], Th. 13	$0 < f \leqslant e < 1$	$\Vert$	[R10], Th. 6	$0 < f < e$	-	[R9], Th. 2

where
(+) means that there are such F and G from the corresponding families of operations that satisfy the left distributivity equation,
(-) means that the distributivity equation is contradictory for any operations F and G from the corresponding families,
($\Vert$) means that for each operations F and G from the corresponding families, the left and the right side of the distributivity equation are incomparable. For the analysis of cases (-) there were used, among others, Theorem 3.2 and Lemma 3.3 in [R1].

In the case of $F \in \mathcal{N}_e$, $G \in \mathcal{N}_f$ we distinguished in [R1] four subcases $F \in \mathcal{N}_e^{\min}$, $G \in \mathcal{N}_f^{\min}$ (Theorem 4.1); $F \in \mathcal{N}_e^{\max}$, $G \in \mathcal{N}_f^{\max}$ (Theorem 4.2); $F \in \mathcal{N}_e^{\min}$, $G \in \mathcal{N}_f^{\max}$ (Theorem 4.5); $F \in \mathcal{N}_e^{\max}$ and $G \in \mathcal{N}_f^{\min}$ (Theorem 4.6) depending on the order between their neutral elements. The sufficient condition in all theorems above was the idempotency of G, where

$$\mathcal{N}_e^{\min} \ni G(x,y) = U^{\min}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [e_2,1]^2 \\ \min(x,y) & \text{elsewhere} \end{cases}, \tag{5}$$

$$\mathcal{N}_e^{\max} \ni G(x,y) = U^{\max}(x,y) = \begin{cases} \min(x,y) & \text{if} (x,y) \in [0,e_2]^2 \\ \max(x,y) & \text{elsewhere} \end{cases}, \tag{6}$$

respectively (see Fig. 1).

Considering inequalities (3) or (4) for operations from different families $F \in \mathcal{N}_e$ and $G \in \mathcal{Z}_k$ we obtained that
in the case when $k > 0$ an operation $F \in \mathcal{N}_e^{\max}$ (Theorem 5.1), in the case when $k < 1$ an operation $F \in \mathcal{N}_e^{\min}$ (Theorem 5.2) and G must be the idempotent nullnorm given by (see Fig. 2)

$$G(x,y) = V_k(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [0,k]^2 \\ \min(x,y) & \text{if } (x,y) \in [k,1]^2 \\ k & \text{if } (x,y) \in D_k \end{cases}. \tag{7}$$

Thus, our considerations in [R1] brought positive results (see Tab. 4), which are a necessary complement of research carried out in [R7] - [R10] and [46, 47].

Table 4: Summarized results from [R1].

F G	V_k			$U_f^{\min}$			$U_f^{\max}$		
	Case	Result	Theorem	Case	Result	Theorem	Case	Result	Theorem
$\mathcal{Z}_s$	-	-	-	$f < s$	$\leqslant$	6.2	$s < f$	$\geqslant$	6.1
$\mathcal{N}_e^{\min}$	$e < k$	$\geqslant$	5.2	$e < f$	$\leqslant$	4.1	$f < e$	$\leqslant$	4.5
$\mathcal{N}_e^{\max}$	$k < e$	$\leqslant$	5.1	$e < f$	$\geqslant$	4.6	$f < e$	$\geqslant$	4.2

Based on Theorems 4.1 and 4.2 in [R1] we have received Corollaries 4.3 and 4.4, constituting an important complement of Propositions 6.2 and 6.6 from the paper [47] indicating that

(i) for $e_1 < e_2$ every uninorm $F \in \mathcal{N}_{e_1}^{\min}$ (triangular conorm $F \in \mathcal{N}_0$) is subdistributive with respect to the idempotent uninorm $G = U^{\min}$ (5),

(ii) for $e_1 > e_2$ every uninorm $F \in \mathcal{N}_{e_1}^{\max}$ (triangular norm $F \in \mathcal{N}_1$) is superdistributive with respect to the idempotent uninorm $G = U^{\max}$ (6).

Moreover, directly from Theorems 4.5 and 4.6 in [R1] we obtained Corollaries 4.7 and 4.8 indicating that

(i) for $e > f$ every uninorm $F \in \mathcal{N}_e^{\min}$ is subdistributive with respect to the idempotent uninorm $G = U^{\max}$ (6),

(ii) for $e < f$ every uninorm $F \in \mathcal{N}_e^{\max}$ is superdistributive with respect to the idempotent uninorm $G = U^{\min}$ (5).

However, in the case of the reverse order of neutral elements we can not get neither subdistributivity nor superdistributivity of these operations even for both idempotent operations, as shown in Example 4.9 in [R1].
As a complement of Propositions 5.2 and 5.5 from the paper [46] we have received Corollaries 5.3 and 5.4 in [R1] indicating that

(i) for $k < e$ every uninorm $F \in \mathcal{N}_e^{\max}$ (triangular norm $F \in \mathcal{N}_1$) is subdistributive with respect to the idempotent nullnorm $G = V_k$ (7),

(ii) for $e < k$ every uninorm $F \in \mathcal{N}_e^{\min}$ (triangular conorm $F \in \mathcal{N}_0$) is superdistributive with respect to the idempotent nullnorm $G = V_k$ (7).

In the converse order, when $F \in \mathcal{Z}_s$ and $G \in \mathcal{N}_f$ (Theorems 6.1 and 6.2 in [R1]), we also received Corollaries 6.3 and 6.4, which complement Proposition 4.1 from the paper [46] indicating that

(i) for $s < f$ every nullnorm $F \in \mathcal{Z}_s$ is superdistributive with respect to the idempotent uninorm $G = U^{\max}$ (6),

(ii) for $f < s$ every nullnorm $F \in \mathcal{Z}_s$ is subdistributive with respect to the idempotent uninorm $G = U^{\min}$ (5).

The problem of (conditional) distributivity for increasing operations with a neutral element

The main results considered in the paper [R2] include characterization of solutions for both conditional and usual distributivity equations between operations $F \in \mathcal{N}_e^{\min} \cup \mathcal{N}_e^{\max}$ and $G \in \mathcal{N}_1 \cup \mathcal{N}_0$, and conversely, what simultaneously generalized results from [54] and extended the set of possible solutions for those already published in [R1, R8, R9, R11].

Since the problem of distributivity for some aggregation operation over a t-conorm (t-norm) gives us only the trivial solution, that is t-conorm (t-norm) in question has to be max (min), it was necessary to restrict the domain of distributivity in the following manner.

Definition 9. Let $F \in \mathcal{N}_e$ with a neutral element $e \in (0,1)$ and $G \in \mathcal{N}_0$ ($G \in \mathcal{N}_1$). We say that an operation F is left conditionally distributive (**LCD**) over an operation G if

$$F(x, G(y,z)) = G(F(x,y), F(x,z)) \quad \text{for} \quad x,y,z \in [0,1] \quad \text{such that} \quad G(y,z) < 1 \quad (G(y,z) > 0).$$

An operation F is right conditionally distributive (**RCD**) over an operation G if

$$F(G(y,z), x) = G(F(y,x), F(z,x)) \quad \text{for} \quad x,y,z \in [0,1] \quad \text{such that} \quad G(y,z) < 1 \quad (G(y,z) > 0).$$

For a commutative operation F (LCD) and (RCD) coincides and are denoted by (CD). This type of distributivity is also known as the restricted distributivity [35] and, although the domain is only weakly restricted, the class of pairs of operators that fulfill (CD) is much wider.

A part of results obtained in [R2], where the neutral element of $F \in \mathcal{N}_e$ was an idempotent element of $G \in \mathcal{N}_0 \cup \mathcal{N}_1$, still gave solutions of the type min, max, wherein for $F \in \mathcal{N}_e^{\min}$ an additional and essential assumption was the continuity of $G \in \mathcal{N}_0$, for $F \in \mathcal{N}_e^{\max}$ the continuity of $G \in \mathcal{N}_1$ and for idempotent $F \in \mathcal{N}_e$ the continuity of $G \in \mathcal{N}_1 \cup \mathcal{N}_0$. They are formulated in Theorems 5.3, 5.5, 5.7, 5.12, 5.13 in [R2], which will be presented jointly.

Theorem 3. *Let* $e \in (0,1)$.
(i) *An operation* $F \in \mathcal{N}_e^{\max}$ *(*$F \in \mathcal{N}_e^{\min}$*) is left or right conditionally distributive over an operation* $G \in \mathcal{N}_0$ *(*$G \in \mathcal{N}_1$*) if and only if* $G = \max$ *(*$G = \min$*).*
(ii) *An operation* $F \in \mathcal{N}_e^{\min}$ *(*$F \in \mathcal{N}_e^{\max}$*) is left or right conditionally distributive over a continuous operation* $G \in \mathcal{N}_0$ *(*$G \in \mathcal{N}_1$*) if and only if* $G = \max$ *(*$G = \min$*).*
(iii) *An idempotent operation* $F \in \mathcal{N}_e$ *is left or right conditionally distributive over a continuous operation* $G \in \mathcal{N}_0$ *(*$G \in \mathcal{N}_1$*) if and only if* $G = \max$ *(*$G = \min$*).*

Non-trivial solutions were obtained in the case when a neutral element e of an operation F was not an idempotent element of an operation G (Theorems 5.8 and 5.15). However, we needed using there an additional assumption of left-continuity of an operation F.

Considerations of the converse assignment of operations F and G i.e. $F \in \mathcal{N}_1 \cup \mathcal{N}_0$ and $G \in \mathcal{N}_e$, no longer required the restricted domain, and concerned usual distributivity.

Theorem 4 ([R2], Theorems 6.4 and 6.8)**.** *Let* $e \in (0,1)$.
(i) *An operation* $F \in \mathcal{N}_0$ *is left or right conditionally distributive over an operation* $G \in \mathcal{N}_e$ *if and only if* F *and* G *have the structure from Fig. 3*
(ii) *An operation* $F \in \mathcal{N}_1$ *is left or right conditionally distributive over an operation* $G \in \mathcal{N}_e$ *if and only if* F *and* G *have the structure from Fig. 4*

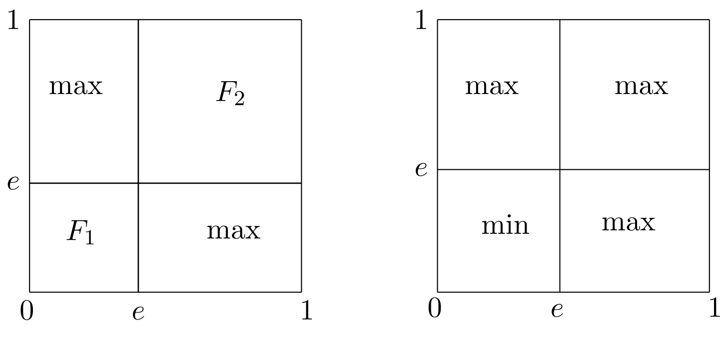

Figure 3: The structure of distributive operations from Theorem 4 (i).

where $(0, F_1, e)$ and $(e, F_2, 1)$ are ordered algebraic structures with distinguished neutral elements e and 1.

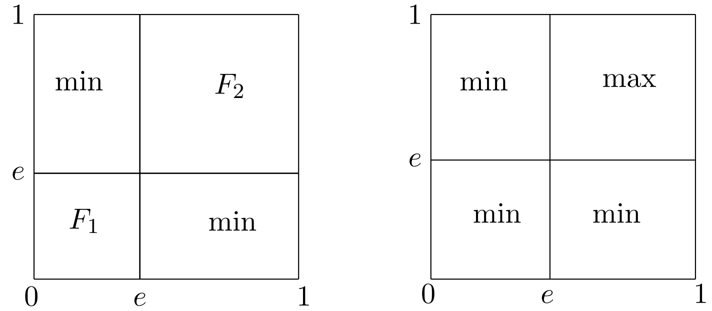

Figure 4: The structure of distributive operations from Theorem 4 (*ii*).

The problem of distributivity for semi-t-operators

The problem of distributivity for t-operators was solved in the paper [46]. But if we leave the assumption of commutativity, the situation becomes much more complex, which requires 24 separate theorems.

Definition 10. An operation $F : [0,1]^2 \to [0,1]$ is called a semi-t-operator if it is associative, increasing with respect to both variables and continuous on the border of the domain
if $F(0,0) = 0$ and $F(1,1) = 1$.
Let $a, b \in [0,1]$. By $\mathcal{F}_{a,b}$ we denote the family of all semi-t-operators such that $F(0,1) = a$, $F(1,0) = b$. In particular, $\mathcal{F}_k := \mathcal{F}_{k,k}$ denotes the set of all associative operations in $\mathcal{Z}_k$.

Theorem 5 ([R3], Theorem 2.12)**.** *$F \in \mathcal{F}_{a,b}$ if and only if there exist associative operations $T \in \mathcal{N}_1$ and $S \in \mathcal{N}_0$ such that*

$$F(x,y) = \begin{cases} aS\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } x, y \in [0,a] \\ b + (1-b)T\left(\frac{x-b}{1-b}, \frac{y-b}{1-b}\right) & \text{if } x, y \in [b,1] \\ a & \text{if } x \leqslant a \leqslant y \\ b & \text{if } y \leqslant b \leqslant x \\ x & \text{elsewhere} \end{cases} \tag{8}$$

for $a \leqslant b$ and

$$F(x,y) = \begin{cases} bS\left(\frac{x}{b}, \frac{y}{b}\right) & \text{if } x, y \in [0,b] \\ a + (1-a)T\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } x, y \in [a,1] \\ a & \text{if } x \leqslant a \leqslant y \\ b & \text{if } y \leqslant b \leqslant x \\ y & \text{elsewhere} \end{cases} \tag{9}$$

for $b \leqslant a$.

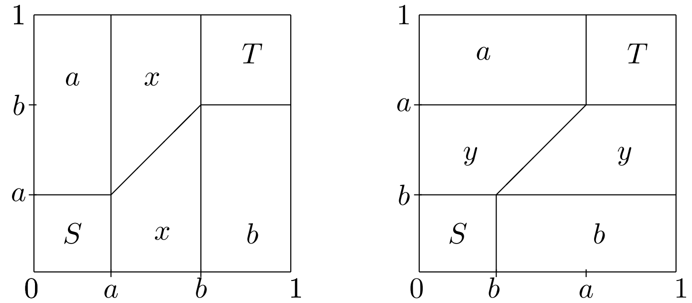

Figure 5: The structure of semi-t-operator F from Theorem 5 (left (8), right (9))

In the paper [R3] (with co-author P. Drygaś) we have characterized the solutions of the left and the right distributivity equations between semi-operators $F \in \mathcal{F}_{a,b}$ and $G \in \mathcal{F}_{c,d}$ (Theorems 4.2 - 4.25), depending on the order between elements a, b of the operation F and elements c, d of the operation G.
The particular feature of the obtained results is that the necessary condition of distributivity is always the idempotency of an operation with respect to which it occurs. Its structure presents the following theorem.

Theorem 6 ([R3], Theorem 2.14)**.** *A semi-t-operator $G \in \mathcal{F}_{c,d}$ is idempotent if and only if it has a structure according to Fig. 6.*

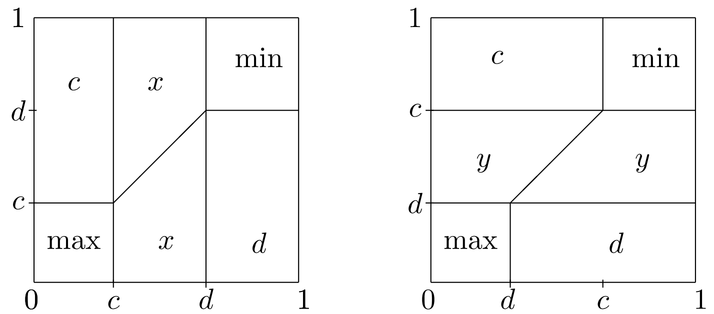

Figure 6: The structure of an idempotent semi-t-operator (left for $c < d$, right for $d < c$).

The sufficient condition in turn extorts the specific structure of the domain of the first operation, whose restrictions have to fulfill additional properties. In addition, we obtained the same solution for both (LD) and (RD) distributivity equations, so in the statement of the following theorems we simply used the term distributivity. The solution of the problem of distributivity for operations from the family of semi-t-operators was carried out comprehensively.

F/G	$G \in \mathcal{F}_{c,d},\ c \leqslant d$		$G \in \mathcal{F}_{c,d},\ d \leqslant c$	
	Case	Result	Case	Result
$F \in \mathcal{F}_{a,b},\ a \leqslant b$	$c < a \leqslant b < d$	Th. 4.2	$d \leqslant c < a \leqslant b$	Th. 4.8
	$c \leqslant d < a \leqslant b$	Th. 4.3	$d < a \leqslant c \leqslant b$	Th. 4.9
	$c < a \leqslant d \leqslant b$	Th. 4.4	$d < a \leqslant b < c$	Th. 4.10
	$a \leqslant c \leqslant b < d$	Th. 4.5	$a \leqslant d \leqslant b \leqslant c$	Th. 4.12
	$a \leqslant b < c \leqslant d$	Th. 4.6	$a \leqslant b < d \leqslant c$	Th. 4.13
$F \in \mathcal{F}_{a,b},\ b \leqslant a$	$c \leqslant d < b \leqslant a$	Th. 4.14	$d \leqslant c < b \leqslant a$	Th. 4.20
	$c < b \leqslant d < a$	Th. 4.15	$d \leqslant b \leqslant c \leqslant a$	Th. 4.21
	$c < b \leqslant a < d$	Th. 4.16	$d < b \leqslant a < c$	Th. 4.22
	$b \leqslant c \leqslant a < d$	Th. 4.18	$b \leqslant d \leqslant a < c$	Th. 4.24
	$b \leqslant a < c \leqslant d$	Th. 4.19	$b \leqslant a < d \leqslant c$	Th. 4.25
$F \in \mathcal{F}_{a,b}$	$a \leqslant c \leqslant d \leqslant b$	Th. 4.7	$a \leqslant d \leqslant c \leqslant b$	Th. 4.11
	$b \leqslant c \leqslant d \leqslant a$	Th. 4.17	$b \leqslant d \leqslant c \leqslant a$	Th. 4.23

In conclusion, inspired by nullnorms and t-operators, we examined functional equations of distributivity in the class of semi-t-operators. We provided a full characterization of solutions of equations (LD) and (RD), i.e. for noncommutative operations $F \in \mathcal{F}_{a,b}$ and $G \in \mathcal{F}_{c,d}$, depending on the order between a, b, c, d. The table above summarizes all the possible cases, which consist of twenty four theorems. It can be observed that in Theorems 4.7, 4.11, 4.17 and 4.23 the both-sided distributivity occurs in the case of the arbitrary operator F. We will present, as an example, two of twenty four achieved results.

Theorem 7 ([R3], Theorem 4.2)**.** *Let $a, b, c, d \in [0,1]$, $c < a \leqslant b < d$. An operation $F \in \mathcal{F}_{a,b}$ is distributive over an operation $G \in \mathcal{F}_{c,d}$ if and only if F and G have the structure as in Fig. 7, where $([0,c], S_1, 0)$, $([c,a], S_2, c)$, $([b,d], T_1, d)$, $([d,1], T_2, 1)$ are ordered algebraic structures with distinguished neutral elements.*

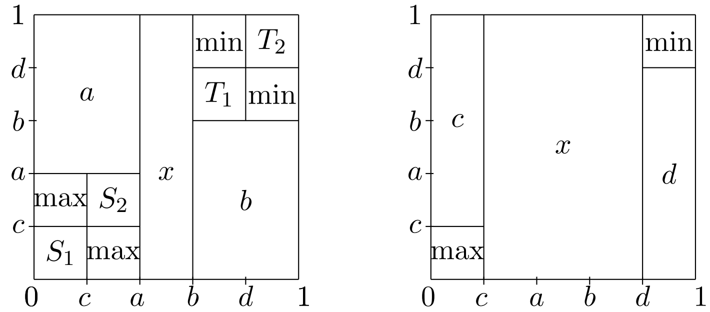

Figure 7: Structures of operations F and G from Theorem 7.

Theorem 8 ([R3], Theorem 4.8)**.** *Let $a, b, c, d \in [0,1]$, $d \leqslant c < a \leqslant b$. An operation $F \in \mathcal{F}_{a,b}$ is distributive over an operation $G \in \mathcal{F}_{c,d}$ if and only if F and G have the structure as in Fig. 8, where $([0,d], S_1, 0)$, $([d,c], S_2, d)$, $([c,a], S_3, c)$, $([b,1], T, 1)$ are ordered algebraic structures with distinguished neutral elements.*

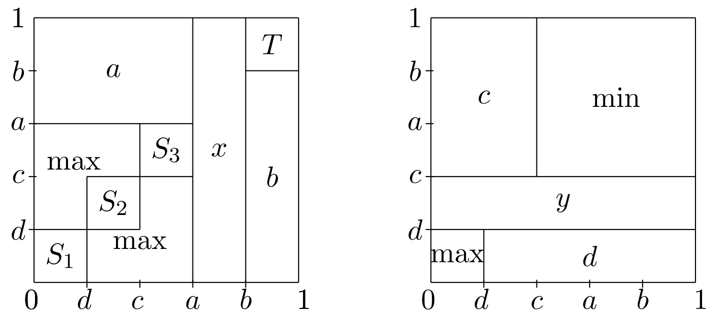

Figure 8: Structures of operations F and G from Theorem 8.

An application of above theorems to nullnorms

Assuming $c = d = k$, we obtained the following theorem.

Theorem 9 ([R3], the associated version of Theorems 5.1 - 5.4)**.** *Let $a, b, k \in [0,1]$.*
(i) For $a < b < k$ an operation $F \in \mathcal{F}_{a,b}$ is distributive over an operation $G \in \mathcal{F}_k$ if and only if G is the idempotent nullnorm (7) *and F has the following form*

$$F(x,y) = \begin{cases} aS\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } x, y \in [0,a] \\ b + (k-b)T_1\left(\frac{x-k}{k-b}, \frac{y-k}{k-b}\right) & \text{if } x, y \in [b,k] \\ k + (1-k)T_2\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } x, y \in [k,1] \\ \min(x,y) & \text{if } b \leqslant \min(x,y) \leqslant k \leqslant \max(x,y) \\ a & \text{if } x \leqslant a \leqslant y \\ b & \text{if } y \leqslant b \leqslant x \\ x & \text{elsewhere} \end{cases},$$

where $([0,a], S, 0)$, $([b,k], T_1, k)$, $([k,1], T_2, 1)$ are ordered algebraic structures with distinguished neutral elements;

(*ii*) *For $k < b < a$ an operation $F \in \mathcal{F}_{a,b}$ is distributive over an operation $G \in \mathcal{F}_k$ if and only if G is the idempotent nullnorm* (7) *and F has the following form*

$$F(x,y) = \begin{cases} kS_1\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } x,y \in [0,k] \\ k + (a-k)S_2\left(\frac{x-k}{a-k}, \frac{y-k}{a-k}\right) & \text{if } x,y \in [k,a] \\ \max(x,y) & \text{if } \min(x,y) \leqslant k \leqslant \max(x,y) \leqslant a \\ b + (1-b)T\left(\frac{x-b}{1-b}, \frac{y-b}{1-b}\right) & \text{if } x,y \in [b,1] \\ a & \text{if } x \leqslant a \leqslant y \\ b & \text{if } y \leqslant b \leqslant x \\ y & \text{elsewhere} \end{cases},$$

where $([0,k], S_1, 0)$, $([k,a], S_2, k)$, $([b,1], T, 1)$ *are ordered algebraic structures with distinguished neutral elements;*

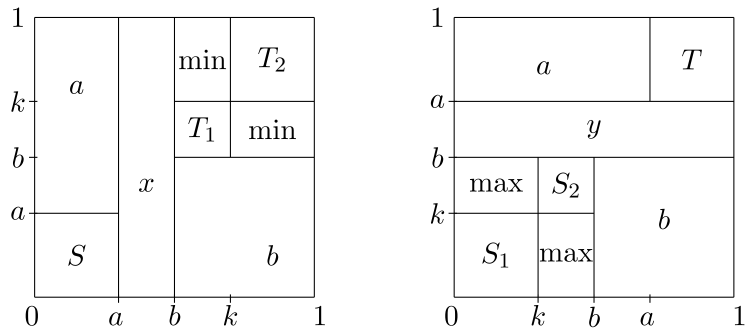

Figure 9: The structure of semi-t-operator F from Theorem 9 (*i*) (left) and (*ii*) (right).

(*iii*) *For $a \leqslant k \leqslant b$ every $F \in \mathcal{F}_{a,b}$ is distributive over $G \in \mathcal{F}_k$ if and only if G is the idempotent t-operator* (7)*;*
(*iv*) *For $b < a < k$ an operation $F \in \mathcal{F}_{a,b}$ is distributive over an operation $G \in \mathcal{F}_k$ if and only if G is the idempotent t-operator* (7) *and F has the following form*

$$F(x,y) = \begin{cases} aS\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } x,y \in [0,a] \\ b + (k-b)T_1\left(\frac{x-k}{k-b}, \frac{y-k}{k-b}\right) & \text{if } x,y \in [b,k] \\ k + (1-k)T_2\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } x,y \in [k,1] \\ \min(x,y) & \text{if } b \leqslant \min(x,y) \leqslant z \leqslant \max(x,y) \\ a & \text{if } x \leqslant a \leqslant y \\ b & \text{if } y \leqslant b \leqslant x \\ y & \text{elsewhere} \end{cases},$$

where $([0,b], S, 0)$, $([a,k], T_1, k)$, $([k,1], T_2, 1)$ *are ordered algebraic structures with distinguished neutral elements.*

The problem of distributivity of semi-t-operators over uninorms

The problem of distributivity for $F \in \mathcal{F}_{a,b}$ over $U \in \mathcal{U}_e$ was solved in the paper [R5], written in cooperation with P. Drygaś and F. Qin. Without the commutativity assumption of semi-t-operators it is still necessary to consider the left and the right distributivity conditions separately. The results of the left distributivity look very similar to those of the right distributivity, but the left distributivity of F and U is considered when $a \leqslant b$, while a right distributivity is considered when $b \leqslant a$. Furthermore, in the case when $a < b$ the operation F has a right neutral element in the subintervals, which add up to the unit interval, and thereby the left distributivity causes the idempotency of a uninorm as in the following lemma.

Lemma 10 ([R5], Lemma 2)**.** *Let $F : X^2 \to X$ have a right (left) neutral element e in a subset $\emptyset \neq Y \subset X$ (i.e. $F(x,e) = x$ ($F(e,x) = x$) for all $x \in Y$). If an operation F is left (right) distributive over an operation $U : X^2 \to X$ satisfying $U(e,e) = e$, then U is idempotent in Y.*

On the other hand, the left neutral element can be obtained only on a subset of the unit interval, which allows only partial results. This can be seen in the following lemma.

Lemma 11 ([R5], Lemma 20)**.** *Let $a, b, e \in [0,1]$. If $a \leqslant b$ and $F \in \mathcal{F}_{a,b}$ is right distributive over $U \in \mathcal{U}_e$, then U is idempotent on the set $[0,a] \cup [b,1]$.*

Hence in the considered paper [R5] the characterization of solutions includes only these cases, where getting the idempotency of a uninorm G by using Lemma 10 was really possible.
Moreover, we can observe that a structure of semi-t-operator is not symmetrical with respect to a diagonal (in contrast to operations from the families of $\mathcal{N}_e$ and $\mathcal{Z}_k$). So, we cannot obtain dual results for the left and the right distributivity conditions, and later combine them to obtain the distributivity how it was possible to do in [R10], in other papers for symmetrical operations, and even unexpectedly in [R3]. In the case of semi-t-operators and uninorms this makes it impossible to apply as it was shown in Example 3 in [R5].

Solutions of the problem of distributivity between a semi-t-operator $F \in \mathcal{F}_{a,b}$ and a uninorm $U \in \mathcal{U}_e$ fulfilling $U(0,1) = 0$ or $U(0,1) = 1$ are contained in Theorems 7-14.
The proof of the necessary condition of each of these theorems is preceded by several lemmas, in which first idempotency of the uninorm (based on Lemma 2 and Theorem 6 using Id-symmetrical function $g : [0,1] \to [0,1]$ with a fixed point e) is shown and next the order of a neutral element of U with respect to a and b is determined, which finally forced a certain division in the structure of semi-t-operator. In turn, the examination of the sufficient condition requires considering a number of cases accordingly depending on the order of a, b and e, and then calculating them meticulously. We will present only one of considered cases in [R5], namely for $F \in \mathcal{F}_{a,b}$, $0 < a \leqslant b$ and U fulfilling $U(0,1) = 0$ (similar results are in the case of a uninorm with $U(0,1) = 1$).

Theorem 12 ([R5], Theorem 8)**.** *Let $a, b, e \in [0,1]$, $0 < a \leqslant b$. An operation $F \in \mathcal{F}_{a,b}$ is left distributive over a uninorm $U \in \mathcal{U}_e$ satisfying $U(0,1) = 0$ if and only if $e < a$, $G = U^{\max}$* (6) *and F has the structure as in Fig. 10 (left), where T is isomorphic with an associative operation from the class $\mathcal{N}_1$, S_1 is isomorphic with an associative operation from the class $\mathcal{N}_0$, e is a right neutral element of $S_2 : [e,a]^2 \to [e,a]$, 0 is a left neutral element of an increasing operation $A : [0,e] \times [e,a] \to [e,a]$ and A, T, S_1, S_2 have common boundary values.*

In the case of the right distributivity equation (RD) between $F \in \mathcal{F}_{a,b}$, where $b \leqslant a$ and a uninorm is under the assumption of $U(0,1) = 0$, we have obtained quite different result to the above. The fulfilment of equation (RD) does not give the same solution as in the case of (LD) for the structure of a semi-t-operator, as it was possible e.g., in [R3].

Theorem 13 ([R5], Theorem 10)**.** *Let $a, b, e \in [0,1]$, $0 < b \leqslant a$. An operation $F \in \mathcal{F}_{a,b}$ is right distributive over a uninorm $U \in \mathcal{U}_e$ satisfying $U(0,1) = 0$ if and only if $e < b$, $G = U^{\max}$* (6) *and F has the structure as in Fig. 10 (right), where T is isomorphic with an associative operation from the class $\mathcal{N}_1$, S_1 is isomorphic with an associative operation from the class $\mathcal{N}_0$, e is a left neutral element of $S_2 : [e,b]^2 \to [e,b]$, 0 is a right neutral element of an increasing operation $B : [e,b] \times [0,e] \to [0,e]$ and B, T, S_1, S_2 have common boundary values.*

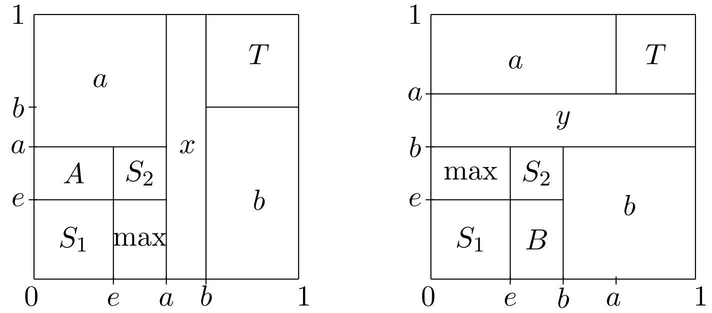

Figure 10: The structure of an operation $F \in \mathcal{F}_{a,b}$ from Theorem 12 (left), from Theorem 13 (right).

The problem of distributivity for 2-uninorms

The problem of distributivity between nullnorms (t-operators) has already been investigated in [46], and with weaker assumptions in [R7]. Let us notice that in the structure of nullnorm V (see Fig. 11) there are two commutative ordered semigroups $([0,k], S_V, 0)$ and $([k,1], T_V, 1)$ with neutral elements 0 and 1.

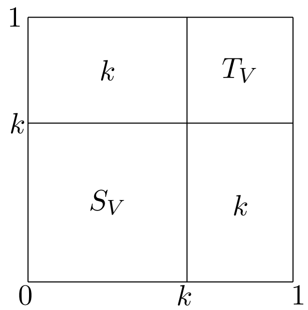

Figure 11: The structure of a nullnorm $V \in \mathcal{V}$.

If now we allow the possibility that neutral elements will be arbitrarily chosen in $[0,k]$ and $[k,1]$, then we obtain a generalization of a nullnorm with the corresponding semigroups isomorphic with uninorms. Therefore, such a generalization has been called a 2-uninorm (cf. [2]). To be more precise

Definition 11 ([R4], Definition 2.11)**.** Let $k \in (0,1)$ and $0 \leqslant e \leqslant k \leqslant f \leqslant 1$. An increasing operation $F : [0,1]^2 \to [0,1]$ is called a 2-uninorm if it is associative, commutative and fulfils

$$\forall_{x \leqslant k}\ F(e,x) = x \quad \text{and} \quad \forall_{x \geqslant k}\ F(f,x) = x. \tag{10}$$

By $\mathcal{U}_{k(e,f)}$ we denote the class of all 2-uninorms.

Directly from (10) and the monotonicity of $F \in \mathcal{U}_{k(e,f)}$ it follows easily that k is a zero element in the interval $[e,f]$ i.e.

$$\forall_{x \in [e,f]}\ F(x,k) = k.$$

Lemma 14 ([R4], Lemma 2.14)**.** *Let $F \in \mathcal{U}_{k(e,f)}$. Then the two mappings U_1, U_2 defined by*

$$U_1(x,y) = \frac{F(kx,ky)}{k} \qquad \textit{for} \ \ x,y \in [0,1],$$

$$U_2(x,y) = \frac{F(k+(1-k)x, k+(1-k)y)}{1-k} \qquad \textit{for} \ \ x,y \in [0,1],$$

are uninorms with neutral elements $\frac{e}{k}$ and $\frac{f-k}{1-k}$, respectively.

Lemma 15 ([R4], Lemma 2.15)**.** *Let* $F \in \mathcal{U}_{k(e,f)}$*. Then we have*

(i) $F(\cdot, 0)$ *is discontinuous at the point* e *if and only if* $U_1(\cdot, 0)$ *is discontinuous at the point* $\frac{e}{k}$*,*

(ii) $F(\cdot, 1)$ *is discontinuous at the point* f *if and only if* $U_2(\cdot, 1)$ *is discontinuous at the point* $\frac{f-k}{1-k}$*.*

Lemma 16 ([R4], Lemma 2.16)**.** *If* $F \in \mathcal{U}_{k(e,f)}$*, then* $F(0,1) \in \{0, k, 1\}$*.*

From the above lemmas we obtain three subclasses of operations in $\mathcal{U}_{k(e,f)}$ based on an element $F(0,1)$, denoted by $\mathcal{C}^0_{k(e,f)}$, $\mathcal{C}^k_{k(e,f)}$, $\mathcal{C}^1_{k(e,f)}$ (or shorter $\mathcal{C}^0, \mathcal{C}^k, \mathcal{C}^1$).

The representation of 2-uninorms $F \in \mathcal{C}^0, \mathcal{C}^k, \mathcal{C}^1$ with the possible points of discontinuity e and f is given by the following theorems ([R4], Theorems 2.17-2.21).

Theorem 17. *Let* $F \in \mathcal{U}_{k(e,f)}$*, where* $F(\cdot, 1)$ *is discontinuous at the points* e *and* f*.* $F(1,k) = k$ *and* $F \in \mathcal{C}^0_k$ *if and only if* $0 < e \leqslant k < f \leqslant 1$ *and* F *has the following form*

$$F = \begin{cases} U^{c_1} & in\ [0,k]^2 \\ U^{c_2} & in\ [k,1]^2 \\ \min & in\ (k,1] \times [0,e) \cup [0,e) \times (k,1] \\ k & in\ [k,1] \times [e,k] \cup [e,k] \times [k,1] \end{cases},$$

where U^{c_1} *and* U^{c_2} *are operations isomorphic with some uninorms from the class* $\mathcal{U}^{\min}_e$ *and* $\mathcal{U}^{\min}_f$*, respectively.*

Theorem 18. *Let* $F \in \mathcal{U}_{k(e,f)}$*, where* $F(\cdot, 1)$ *is discontinuous at the point* e *and* $F(\cdot, e)$ *is discontinuous at the point* f*.* $F(1,k) = 1$ *and* $F \in \mathcal{C}^0_1$ *if and only if* $0 < e \leqslant k \leqslant f < 1$ *and*

$$F = \begin{cases} U^{c} & in\ [0,k]^2 \\ U^{d} & in\ [k,1]^2 \\ \min & in\ (k,1] \times [0,e) \cup [0,e) \times (k,1] \\ \max & in\ (f,1] \times [e,k) \cup [e,k) \times (f,1] \\ k & in\ [k,f] \times [e,k] \cup [e,k] \times [k,f] \end{cases},$$

where U^c *and* U^d *are operations isomorphic with some uninorms from the class* $\mathcal{U}^{\min}_e$ *and* $\mathcal{U}^{\max}_f$*, respectively.*

Theorem 19. *Let* $F \in \mathcal{U}_{k(e,f)}$*, where* $F(\cdot, 0)$ *is discontinuous at the points* e *and* f*.* $F(0,k) = k$ *and* $F \in \mathcal{C}^1_k$ *if and only if* $0 \leqslant e < k \leqslant f < 1$ F *has the following form*

$$F = \begin{cases} U^{d_1} & in\ [0,k]^2 \\ U^{d_2} & in\ [k,1]^2 \\ \max & in\ (f,1] \times [0,k) \cup [0,k) \times (f,1] \\ k & in\ [k,f] \times [0,k] \cup [0,k] \times [k,f] \end{cases},$$

where U^{d_1} *and* U^{d_2} *are operations isomorphic with some uninorms from the class* $\mathcal{U}^{\max}_e$ *and* $\mathcal{U}^{\max}_f$*, respectively.*

Theorem 20. *Let* $F \in \mathcal{U}_{k(e,f)}$*, where* $F(\cdot, f)$ *is discontinuous at the point* e *and* $F(\cdot, 0)$ *is discontinuous at the point* f*.* $F(0,k) = 0$ *and* $F \in \mathcal{C}^1_0$ *if and only if* $0 < e \leqslant k \leqslant f < 1$ *and*

$$F = \begin{cases} U^{c} & in\ [0,k]^2 \\ U^{d} & in\ [k,1]^2 \\ \min & in\ (k,f] \times [0,e) \cup [0,e) \times (k,f] \\ \max & in\ (f,1] \times [0,k) \cup [0,k) \times (f,1] \\ k & in\ [k,f] \times [e,k] \cup [e,k] \times [k,f] \end{cases},$$

where U^c *and* U^d *are operations isomorphic with some uninorms from the class* $\mathcal{U}^{\min}_e$ *and* $\mathcal{U}^{\max}_f$*, respectively.*

Theorem 21. *Let $F \in \mathcal{U}_{k(e,f)}$, where $F(\cdot,0)$ is discontinuous at the point $e \in (0,k]$ and $F(\cdot,1)$ is discontinuous at the point $f \in [k,1)$. Then $F \in \mathcal{C}^k$ if and only if $0 \leqslant e < k < f \leqslant 1$ and F has the following form*

$$F = \begin{cases} U^d & in\ [0,k]^2 \\ U^c & in\ [k,1]^2 \\ k & in\ [k,1] \times [0,k] \cup [0,k] \times [k,1] \end{cases},$$

where U^d and U^c are operations isomorphic with some uninorms from the class $\mathcal{U}_e^{\max}$ and $\mathcal{U}_f^{\min}$, respectively.

In the paper [R4] (with co-author P. Drygaś) the problem of distributivity between operations $F \in \mathcal{U}_{k_1(e_1,f_1)}$ and $G \in \mathcal{U}_{k_2(e_2,f_2)}$ has been solved, distinguishing both the order of their zero elements as well as their specific structures resulting from the classification. This required consideration of five cases, in which essential is the form of the operation F and the proper order for respective neutral elements that distributivity could occur. It turned out that the full characterization depends on twenty five cases which makes this problem even more interesting. The main results of [R4] are Theorems 5.1, 5.3, 5.5, 5.7 and appropriate Remarks 5.2, 5.4, 5.6, 5.8.
Here, as in the previously discussed papers, the necessary condition of distributivity between two operations is the idempotency of that one over which it occurs. The proof of such an idempotency and the final structure of G is not conventional. It usually consists of several steps, one of which involves the possible reduction of elements e_2 and f_2 of this operation to 0, k and/or 1. Then in certain subclasses of 2-uninorms it makes a significant simplification of the initial structure of this operation. Let us see, as an example, the following theorem.

Theorem 22 ([R4], Theorem 5.1)**.** *Let $k_1, k_2 \in [0,1]$ and $k_2 \leqslant k_1$. A 2-uninorm $F \in \mathcal{C}^{k_1}_{(e_1,f_1)}$ is distributive over a 2-uninorm $G \in \mathcal{U}_{k_2(e_2,f_2)}$ if and only if $e_1 \leqslant e_2 \leqslant k_2 \leqslant k_1 \leqslant f_2 \leqslant f_1$, G is idempotent given by*

$$\mathcal{C}^{k_2} \ni G = \begin{cases} U^{\max} & in\ [0,k_2]^2 \\ U^{\min} & in\ [k_2,1]^2 \\ k_2 & elsewhere \end{cases}, \qquad \mathcal{C}^0_{k_2} \ni G = \begin{cases} U^{\min} & in\ [k_2,1]^2 \\ \min & elsewhere \end{cases}, \tag{11}$$

$$\mathcal{C}^1_{k_2} \ni G = \begin{cases} U^{\max} & in\ [0,k_2]^2 \\ \max & elsewhere \end{cases}, \qquad \mathcal{C}^1_0 \ni G = U^{\min}\ (see (5)), \quad \mathcal{C}^0_1 \ni G = U^{\max}\ (see (6)) \tag{12}$$

respectively, and F has the following form

$$F = \begin{cases} T_{U^d} & in\ [0,e_1]^2 \\ S^1_{U^d} & in\ [e_1,k_2]^2 \\ S^2_{U^d} & in\ [k_2,k_1]^2 \\ U^c & in\ [k_1,1]^2 \\ k_1 & in\ [k_1,1] \times [0,k_1] \cup [0,k_1] \times [k_1,1] \\ \max & elsewhere \end{cases}, \tag{13}$$

where $([0,e_1], T_{U^d}, e_1)$, $([e_1,k_2], S^1_{U^d}, e_1)$ and $([k_2,k_1], S^2_{U^d}, k_2)$ are commutative ordered semigroups with neutral elements equal to e_1, e_1 and k_2, respectively.

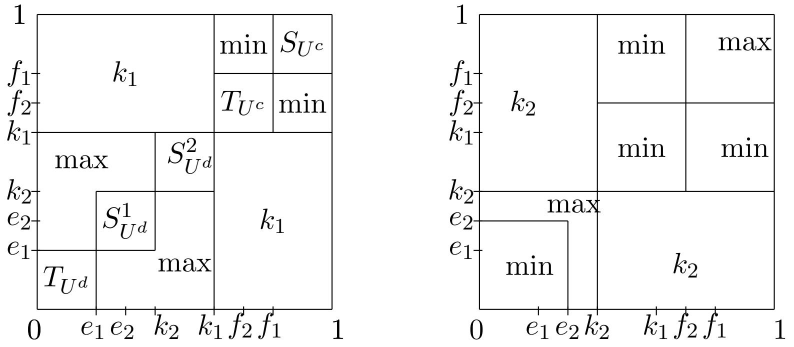

Figure 12: Structures of 2-uninorms from Theorem 22 for $F \in \mathcal{C}^{k_1}$ and $G \in \mathcal{C}^{k_2}$.

Remark 2 ([R4], Remark 5.2). In the case of distributivity between $F \in \mathcal{C}^{k_1}_{(e_1,f_1)}$ and $G \in \mathcal{U}_{k_2(e_2,f_2)}$, where $k_1 < k_2$, the only difference is in the form of a 2-uninorm F, namely

$$F = \begin{cases} U^d & \textit{in } [0,k_1]^2 \\ T^1_{U^c} & \textit{in } [k_1,k_2]^2 \\ T^2_{U^c} & \textit{in } [k_2,f_1]^2 \\ S_{U^c} & \textit{in } [f_1,1]^2 \\ k_1 & \textit{in } [k_1,1]\times[0,k_1]\cup[0,k_1]\times[k_1,1] \\ \min & \textit{elsewhere} \end{cases},$$

where $([k_1,k_2],T^1_{U^c},k_2)$, $([k_2,f_1],T^2_{U^c},f_1)$ and $([f_1,1],S_{U^c},f_1)$ are commutative ordered semigroups with neutral elements equal to k_2, f_1 and f_1, respectively.

In the case of solutions of distributivity equation for 2-uninorms from subclasses $\mathcal{C}^0_1$ and $\mathcal{C}^1_0$ with respect to any 2-uninorm, the obtained idempotent structures overlap (are the same). Combining both cases we received the following result.

Theorem 23 ([R4], Theorem 5.7)**.** *Let $k_1,k_2 \in [0,1]$ and $k_2 \leqslant k_1$. A 2-uninorm $F \in \{\mathcal{C}^0_1,\mathcal{C}^1_0\}$ is distributive over a 2-uninorm $G \in \mathcal{U}_{k_2(e_2,f_2)}$ if and only if $e_2 \leqslant e_1 \leqslant k_1 \leqslant k_2 \leqslant f_2 \leqslant f_1$, G is idempotent given by*
$\mathcal{C}^{k_2} \ni G = V_k$ (see (7)),

$$\mathcal{C}^0_{k_2} \ni G = \begin{cases} U^{\min} & \textit{in } [0,k_2]^2 \\ k_2 & \textit{in } [e_2,k_2]\times[k_2,1]\cup[k_2,1]\times[e_2,k_2] \\ \min & \textit{elsewhere} \end{cases}, \quad \mathcal{C}^1_{k_2} \ni G = \begin{cases} U^{\max} & \textit{in } [k_2,1]^2 \\ k_2 & \textit{in } (f_2,1]\times[0,k_2)\cup[0,k_2)\times(f_2,1] \\ \max & \textit{elsewhere} \end{cases},$$

$$\mathcal{C}^0_1 \ni G = \begin{cases} U^{\min} & \textit{in } [0,k_2]^2 \\ U^{\max} & \textit{in } [k_2,1]^2 \\ k_1 & \textit{in } [k_1,f_1]\times[e_1,k_1]\cup[e_1,k_1]\times[k_1,f_1] \\ \min & \textit{in } (k_2,1]\times[0,e_2)\cup[0,e_2)\times(k_2,1] \\ \max & \textit{elsewhere} \end{cases}, \quad \mathcal{C}^1_0 \ni G = \begin{cases} U^{\min} & \textit{in } [0,k_2]^2 \\ U^{\max} & \textit{in } [k_2,1]^2 \\ k_1 & \textit{in } [k_1,f_1]\times[e_1,k_1]\cup[e_1,k_1]\times[k_1,f_1] \\ \min & \textit{in } (k_1,f_1]\times[0,e_1)\cup[0,e_1)\times(k_1,f_1] \\ \max & \textit{elsewhere} \end{cases}$$

respectively, and F has the following form

$$\mathcal{C}^0_1 \ni F = \begin{cases} T_{U^c} & \textit{in } [0,e_1]^2 \\ S^1_{U^c} & \textit{in } [e_1,k_2]^2 \\ S^2_{U^c} & \textit{in } [k_2,k_1]^2 \\ U^d & \textit{in } [k_1,1]^2 \\ k_1 & \textit{in } [k_1,f_1]\times[e_1,k_1]\cup[e_1,k_1]\times[k_1,f_1] \\ \min & \textit{in } (k_1,1]\times[0,e_1)\cup[0,e_1)\times(k_1,1] \\ \max & \textit{elsewhere} \end{cases}, \quad \mathcal{C}^1_0 \ni F = \begin{cases} T_{U^c} & \textit{in } [0,e_1]^2 \\ S^1_{U^c} & \textit{in } [e_1,k_2]^2 \\ S^2_{U^c} & \textit{in } [k_2,k_1]^2 \\ U^d & \textit{in } [k_1,1]^2 \\ k_1 & \textit{in } [k_1,f_1]\times[e_1,k_1]\cup[e_1,k_1]\times[k_1,f_1] \\ \min & \textit{in } (k_1,f_1]\times[0,e_1)\cup[0,e_1)\times(k_1,f_1] \\ \max & \textit{elsewhere} \end{cases},$$

where ($[0,e_1],T_{U^c},e_1$), ($[e_1,k_2],S^1_{U^c},e_1$) and ($[k_2,k_1],S^2_{U^c},k_2$) are commutative ordered semigroups with neutral elements equal to e_1, e_1 and k_1, respectively.

Remark 3 ([R4], Remark 5.8). In the case of $k_1 < k_2$ we have

$$\mathcal{C}_1^0 \ni F = \begin{cases} U^c & in\ [0,k_1]^2 \\ T^1_{U^d} & in\ [k_1,k_2]^2 \\ T^2_{U^d} & in\ [k_2,f_1]^2 \\ S_{U^d} & in\ [f_1,1]^2 \\ k_1 & in\ [k_1,f_1]\times[e_1,k_1]\cup[e_1,k_1]\times[k_1,f_1] \\ \max & in\ (f_1,1]\times[e_1,k_1)\cup[e_1,k_1)\times(f_1,1] \\ \min & elsewhere \end{cases}, \quad \mathcal{C}_0^1 \ni F = \begin{cases} U^c & in\ [0,k_1]^2 \\ T^1_{U^d} & in\ [k_1,k_2]^2 \\ T^2_{U^d} & in\ [k_2,f_1]^2 \\ S_{U^d} & in\ [f_1,1]^2 \\ k_1 & in\ [k_1,f_1]\times[e_1,k_1]\cup[e_1,k_1]\times[k_1,f_1] \\ \max & in\ (f_1,1]\times[0,k_1)\cup[0,k_1)\times(f_1,1] \\ \min & elsewhere \end{cases},$$

where $([k_1,k_2],T^1_{U^d},k_2)$, $([k_2,f_1],T^2_{U^d},f_1)$ and $([f_1,1],S_{U^d},f_1)$ are commutative ordered semigroups with neutral elements equal to k_2, f_1 and f_1, respectively.

The problem of modularity for 2-uninorms

In the paper [R6] (with co-authors W. Fechner and L. Zedam) the problem of modularity between operations from the class of 2-uninorms has been resolved. In particular, for these operations, depending on the position of their neutral elements and having the same zero element, we obtained both positive and negative results. They are analogous to the results of the general case for operators with different zero elements.

The modularity equation for aggregation operations $F \in \mathcal{U}_{k_1(e_1,f_1)}$ and $G \in \mathcal{U}_{k_2(e_2,f_2)}$ in a particular case when $k_1 = k_2 = k \notin \{0,1\}$ extorted their specific structures. In twenty investigated cases, we distinguished forms of both operations and the proper order of the respective neutral elements. The ordering obtained is presented in the following lemma.

Lemma 24 ([R6], Lemma 6.1). *If $k_1 = k_2 = k \in (0,1)$ and $F \in \mathcal{U}_{k_1(e_1,f_1)}$, where $0 \leqslant e_1 \leqslant k_1 \leqslant f_1 \leqslant 1$ is modular over $G \in \mathcal{U}_{k_2(e_2,f_2)}$, $0 \leqslant e_2 \leqslant k_2 \leqslant f_2 \leqslant 1$, then*

$$0 \leqslant e_2 \leqslant e_1 \leqslant k \leqslant f_2 \leqslant f_1 \leqslant 1.$$

In turn, if $e_1 = e_2 = e$ and $f_1 = f_2 = f$, then using twice the theorem (Theorem 5.2 in [R6]), which says that if two commutative operations having the same neutral element are modular, then they are equal in terms of structure, we get that $F|_{[0,k]^2} = G|_{[0,k]^2}$ and $F|_{[k,1]^2} = G|_{[k,1]^2}$ simultaneously.

Thus, by considering the structure of operations for each subclass of 2-uninorms, we could immediately conclude that the only possibility is to consider the modularity between 2-uninorms F and G from the same subclasses. In this case, it turned out that they overlapped in the whole unit square, which also reduced considerations to automodularity or more specifically, to the associativity equation on the restricted domain i.e.

$$F(x,F(y,z)) = F(F(x,y),z) \text{ for all } x,y,z \in [0,1] \text{ such that } z \leqslant x.$$

If $e_2 < e_1$ and $f_1 = f_2 = f$, then $U_{f_1} = U_{f_2} = U_f$ in the square $[k,1]^2$, for which consideration of the modularity equation for operations from the respective subclasses of 2-uninorms is possible only if

(i) $F,G \in \mathcal{C}^k \cup \mathcal{C}^0_k$, $F,G \in \mathcal{C}^k$, $F,G \in \mathcal{C}^1_k$,

(ii) $F \in \mathcal{C}^0_1$ and $G \in \mathcal{C}^1_0 \cup \mathcal{C}^1_k$,

(iii) $F \in \mathcal{C}^1_k$ and $G \in \mathcal{C}^0_1 \cup \mathcal{C}^1_0$,

(iv) $F \in \mathcal{C}^1_0$ and $G \in \mathcal{C}^1_k \cup \mathcal{C}^0_1$.

If $e_1 = e_2 = e$ and $f_2 < f_1$, then $U_{e_1} = U_{e_2} = U_e$ in the square $[0,k]^2$, for which consideration of the modularity equation for operations from the respective subclasses of 2-uninorms is possible only if *(i)* $F, G \in \mathcal{C}^k \cup \mathcal{C}_k^1$, $F, G \in \mathcal{C}^k$, $F, G \in \mathcal{C}_k^0$,

(ii) $F \in \mathcal{C}_1^0$ and $G \in \mathcal{C}_k^0 \cup \mathcal{C}_0^1$,

(iii) $F \in \mathcal{C}_k^0$ and $G \in \mathcal{C}_1^0 \cup \mathcal{C}_0^1$,

(iv) $F \in \mathcal{C}_0^1$ and $G \in \mathcal{C}_1^0 \cup \mathcal{C}_k^0$.

According to the proposed cases we received both negative and positive results. Main results of [R6] concerning the modularity equation of 2-uninorms for which we received non-trivial solutions are Theorems 6.4 - 6.7 for $e_2 < e_1$ and $f_1 = f_2 = f$ and Theorems 6.8 - 6.11 for $e_1 = e_2 = e$ and $f_2 < f_1$. More precisely, considering, for example, the case of $F, G \in \mathcal{C}_k^0$, for which $0 < e_2 = e_1 = e \leqslant k < f_2 < f_1 \leqslant 1$ we have proved the following theorem (by using Theorem 3.4 *(iii)* in [R6] for $F_{|[k,1]^2}$ and $G_{|[k,1]^2}$).

Theorem 25 ([R6], Theorem 6.10)**.** *Let $F, G \in \mathcal{C}_k^0$, where $0 < e = e_2 = e_1 \leqslant k < f_2 < f_1 \leqslant 1$. F is modular over G if and only if these 2-uninorms have the following structures*

$$F = \begin{cases} U^c & \textit{in } [0,k]^2 \\ T & \textit{in } [k,f_2]^2 \\ k & \textit{in } [k,1]\times[e,k]\cup[e,k]\times[k,1] \\ \min & \textit{elsewhere} \end{cases}, \qquad G = \begin{cases} U^c & \textit{in } [0,k]^2 \\ T & \textit{in } [k,f_2]^2 \\ \max & \textit{in } [f_2,1]^2 \\ k & \textit{in } [k,1]\times[e,k]\cup[e,k]\times[k,1] \\ \min & \textit{elsewhere} \end{cases},$$

where $U^c : [0,k]^2 \to [0,k]$ $\mathcal{U}_e^{\min}$ and $T : [k,f_2]^2 \to [k,f_2]$ are operations isomorphic with some uninorm from the class $\mathcal{U}_e^{\max}$ and triangular norm, respectively.

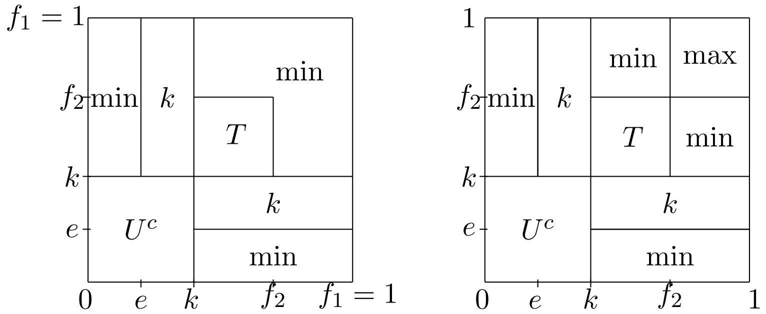

Figure 13: The structure of operations $F, G \in \mathcal{C}_k^0$ from Theorem 25.

The results obtained in [R6] that are consistent with cases $e_2 < e_1$, $f_1 = f_2 = f$ and $e_1 = e_2 = e$, $f_2 < f_1$, where the modularity equation has no solutions, are collected in Theorems 6.12 - 6.20 including interesting examples in Remarks 6.14 and 6.19.

In the table below, I set all results obtained in [R6] concerning the modularity equation between 2-uninorms from each of defined subclasses where: (+) means there are solutions, (-) means the lack of solutions and ($\sharp$) means there is a contradiction between operations parameters.

$F \setminus G$	$\mathcal{C}_k^0$	$\mathcal{C}_1^0$	$\mathcal{C}_k^1$	$\mathcal{C}_0^1$	$\mathcal{C}^k$
$\mathcal{C}_k^0$	(+) Th. 6.10	(+) Th. 6.11	($\sharp$)	(-) Remark 6.19	(+) Th. 6.4
$\mathcal{C}_1^0$	(-) Th. 6.18	($\sharp$)	(-) Remark 6.14	(-) Th. 6.13 and 6.18	($\sharp$)
$\mathcal{C}_k^1$	($\sharp$)	(-) Th. 6.15	(+) Th. 6.6	(-) Th. 6.15	(-) Th. 6.17
$\mathcal{C}_0^1$	(-) Th. 6.20	(-) Th. 6.16 and 6.20	(+) Th. 6.7	($\sharp$)	($\sharp$)
$\mathcal{C}^k$	(-) Th. 6.12	($\sharp$)	(+) Th. 6.8	($\sharp$)	(+) Th. 6.5 and 6.9

I would like to point out that only a partial characterization of the solutions of equation (1) has been obtained so far, that is, in the case where both 2-uninorms F and G have the same distinguished element k. However, the results involving the modularity between 2-uninorms in the general case, where the order between zero elements and neutral elements is unknown, are Theorem 6.21 in [R6] and the result here below.

Theorem 26 ([R6], Theorem 6.22)**.** *Let $F \in \mathcal{C}_0^1$, where $0 < e_1 \leqslant k_1 \leqslant f_1 < 1$ and $G \in \mathcal{C}_k^0 \cup \mathcal{C}_1^0$, where $0 < e_2 \leqslant k_2 < f_2 < 1$. Then F is never modular over G i.e. the equation* (1) *has no solutions.*

Seeking solutions of the functional equation is strictly related to the specific structure of functions considered in it. As it might be expected, we obtained negative results in most of the cases examined. Indeed, this indicates the power of the modularity condition and how difficult it is to fulfill it in the case of functions with a more complex structure.

5. A summary of the remaining scientific achievements.

5.1 List of other published scientific papers.

5.1.1 Scientific papers in journals indexed in the database Journal Citation Reports (2016 Edition):

[R7] E. Rak, Distributivity equation for nullnorms, Journal of Electrical Engineering 56, 12/s (2005), 53–55.

[R8] **E. Rak**, P. Drygaś, Distributivity between uninorms, Journal of Electrical Engineering 57, 7/s (2006), 35–38.

[R9] E. Rak, Some remarks about distributivity between uninorms, Journal of Electrical Engineering 58, 7/s (2007), 41-42.

[R10] J. Drewniak, P. Drygaś, **E. Rak**, Distributivity equations for uninorms and nullnorms, Fuzzy Sets and Systems 159 (2008), 1646-1657.

[R11] J. Drewniak, **E. Rak**, Subdistributivity and superdistributivty of binary operations, Fuzzy Sets and Systems 161 (2010), 189-210.

[R12] R. Rak, **E. Rak**, Route to chaos in generalized logistic map, Acta Physica Polonica A 127 (2015), 113-117.

[R13] K.N. Agbeko, W. Fechner, **E. Rak**, On lattice-valued maps stemming from the notion of optimal average, Acta Mathematica Hungarica 152 (1) (2017), 72–83.

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A description of the scientific output of the habilitation candidate obtained before and after the doctoral degree and which is not included in the set of publications of scientific achievements

Papers [R7] - [R10] as well as [R14]-[R15] were published before the doctoral degree.
These include the full characterization of pairs of increasing binary operations with a distinguished idempotent element (i.e. neutral element or zero element) that fulfill the distributivity axiom under possibly weak assumptions. The particular feature of obtained results is that the necessary condition of distributivity is always the idempotency of an operation with respect to which it occurs, while the sufficient condition forces a certain structure of the domain of the first operation, whose restrictions must fulfill additional properties.

Referring to results of [46]-[47], on my own in [R7], [R9] and in cooperation with P. Drygaś and J. Drewniak in [R8], [R10], we proved the superfluity of certain assumptions imposed on operations that were subjected to checking that they meet the distributivity equation. It turned out that both the associativity and the commutativity is not necessary in such considerations. Consequently, were defined two classes of increasing operations generalizing uninorms and nullnorms, respectively.

Let us emphasize that the paper [R10], which is devoted to the distributivity equation of generalized uninorms and nullnorms, and vice versa, is the most frequently cited paper of the habilitation candidate and presently has 41 citations (without self-citations) according to the Google Scholar database.

The paper [R15] refers indirectly to the above-described solutions of the distributivity equation for operations from the family of $\mathcal{N}_e$. We indicated in it the minimum set of assumptions to be applied to $\mathcal{N}_e$ to derive its idempotency. In this way, we have proved Theorem 3.3, which states that an operation from the class $\mathcal{N}_e^{\min}$ or $\mathcal{N}_e^{\max}$ is superidempotent in $[0,e]$ and subidempotent in $[e,1]$ if and only if it is the only (smallest) idempotent uninorm from the class $\mathcal{N}_e^{\min}$ or the only (largest) idempotent uninorm from the class $\mathcal{N}_e^{\max}$. In addition, we gave examples of the significance of all imposed assumptions (see Example 3.1).

Other papers (published after the doctoral degree) can be divided broadly into two groups. The first the most numerous of them, constitute publications [R11],[R16]-[R19],[R23] still centered around the functional equations and inequalities.

The joint paper [R11] with J. Drewniak, is essentially a detailed summary of different results regarding the autodistributivity ($F = G$ in equation (LD)) and distributivity inequalities of a class of common binary operations like means, triangular norms and triangular conorms. Attempts to their appropriate generalization, however, led to the construction of various counterexamples. Hence most of these results concern the well-known examples of the mentioned operations. Through accurate computation of all cases included in Tables 3 and 4 in [R11], we managed to point out some errors found in preceding papers regarding this topic (formulas (64),(65) in [50] (see Example 3.5 in [R11]) or Theorems 6 and 10 in [12] (see comment to Corollary 3.13 in [R11]).

In the paper [R11] some of dependencies between the property of domination and the subdistributivity or the superdistributivity have been also shown . Domination is a property of operations which plays an important role in considerations connected with the distributivity functional inequalities. Schweizer and Sklar [60] introduced the notion of domination between associative binary operations with a common domain and a common neutral element.

Definition 12 (cf. [60], Definition 12.7.2)**.** Let $F, G : [0,1]^2 \to [0,1]$. We say that an operation F dominates an operation G ($F \gg G$) if

$$F(G(x,y), G(z,w)) \geqslant G(F(x,z), F(y,w)) \quad \text{for all } x,y,z,w \in [0,1].$$

Two main theorems in [R11] describing the relation between domination and functional distributivity inequalities for binary operations under the minimal set of assumptions are as follows.

Theorem 27 ([R11], Theorem 5.3)**.** *Let $F, G : [0,1]^2 \to [0,1]$ and G be an increasing operation.*
- *If F is left and right superdistributive with respect to G and $G \geqslant \max$, then F dominates G.*
- *If F is left and right subdistributive with respect to G and $G \leqslant \min$, then G dominates F.*

Theorem 28 ([R11], Theorem 5.4)**.** *Let $F, G : [0,1]^2 \to [0,1]$ and F be an increasing operation.*
- *If F dominates G, which is subidempotent ($G(x,x) \leqslant x$), then F is superdistributive with respect to G.*
- *If G dominates F, which is superidempotent ($G(x,x) \geqslant x$), then F is subdistributive with respect to G.*

The first one shows directly that by the straight chain of inequalities- the left and the right superdistributivity F with respect to $G \geqslant \max$ implies the dominance of F over G, while the second would seem the converse theorem (although it is not) replacing $G \geqslant \max$ by the subidempotency of G.

In the broader sense, more general study on dominance relation for binary operations defined in partially ordered sets, also with respect to distributivity inequalities, includes [R23], written jointly with U. Bentkowska, J. Drewniak, P. Drygaś, and A. Król.

Papers [R16], [R17], [R18] published in conference proceedings mainly include specific cases of the corresponding ones [R2], [R6], [R5], discussed in detail in Section 4 of this summary of professional accomplishments.

The results in [R19], written jointly with P. Drygaś and L. Zedam, are a generalization of the results from the paper [R4]. There was introduced the concept of the class of 2-semi-uninorms $\mathcal{N}_{k(e,f)}$, together with the structuring of one of its possible subclasses $\mathcal{N}_{k(e,f)}^k$. Further we characterized the solutions of distributivity equations (LD) and (RD) for operations of this subclass, taking into account the order of their zero elements and neutral elements. The obtained solutions including

Theorems 5.1 and 5.3 and Remark 5.4 in [R19] have the following graphical representation (see Fig. 14, 15 and 16).

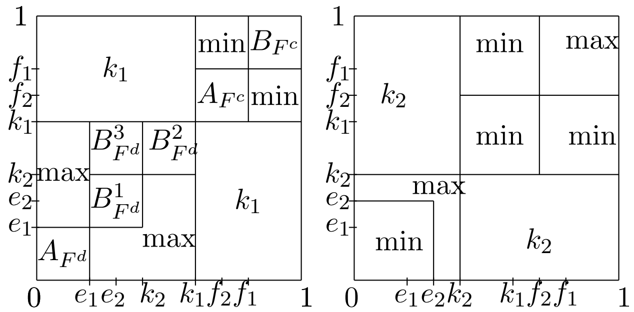

Figure 14: The structure of $F \in \mathbf{N}^{k_1}_{k_1(e_1,f_1)}$ and $G \in \mathbf{N}^{k_2}_{k_2(e_2,f_2)}$ from Th. 5.1 in [R19], when $k_2 \leqslant k_1$.

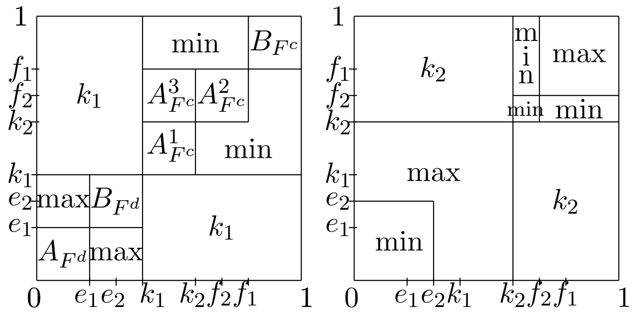

Figure 15: The structure of $F \in \mathbf{N}^{k_1}_{k_1(e_1,f_1)}$ and $G \in \mathbf{N}^{k_2}_{k_2(e_2,f_2)}$ from Th. 5.3 in [R19], when $k_1 < k_2$.

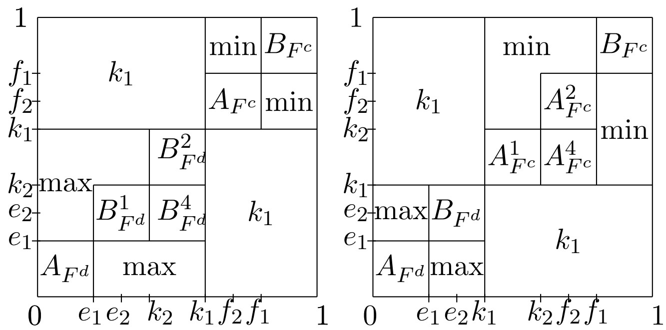

Figure 16: The structure of $F \in \mathbf{N}^{k_1}_{k_1(e_1,f_1)}$ from Remark 5.4 in [R19] (case $k_2 \leqslant k_1$ (left), case $k_1 < k_2$ (right).

The second group of publications (after PhD) are papers [R12]-[R13] and [R20]-[R22]. In the paper [R12], with co-author R. Rak, a generalization of logistic mapping, well-known in dynamical systems theory was proposed. The obtained difference equation is the following $x_{n+1} = f_r(x_n) = r^p x_n(1-x_n^q),\ \ x_n \in [0,1],\ n = 0,1,2,...$, where p and q are any positive real values. For the proposed generalized equation a detailed analysis and the character of the Feigenbaum model of transition from regularity to chaos for the whole spectrum of model parameters was presented. In particular, for $p = q$ the value of the $r_{\max}$ parameter, which is responsible for the dynamics of the system, is the same as for the classical logistic map $r_max = 4$. As an example, we carried out analytical and quantitative analysis in the case where $p = 1$ and $q = 2$, both in the periodic and chaotic regime where the Lyapunov exponent has positive values. It turned out that the dynamics of this equation is faster than in the case of the logistic map — the value of the parameter $r_{\max} \approx 2,3$.

For chaotic area we have found an analytical form of the invariant density function. For both regimes we have proposed specific form of data representation which allowed to obtain a non-trivial structure of an attractor set.

In the paper [R13] (joint work with K.N. Agbeko and W. Fechner) certain lattice-valued maps through associated functional equations and inequalities were considered. We dealt with morphisms between an algebraic structure and an ordered structure i.e. the following functional equation

$$T(x * y) = T(x) \vee T(y), \quad x, y \in S \tag{14}$$

and its related functional inequalities

$$T(x * y) \geqslant T(x) \vee T(y), \quad x, y \in S, \tag{15}$$

$$T(x * y) \leqslant T(x) \vee T(y), \quad x, y \in S, \tag{16}$$

where $(S, *)$ is a semigroup and $(L, \leqslant)$ is a lattice.

Main results in [R13] essentially involved the topic of separation (Theorems 3.1 and 3.2) and the Hyers-Ulam-type stability problem (Theorem 3.4).

Separation theorems have been studied by several authors. A classical result is the Mazur-Orlicz Theorem [48], which was generalized by R. Kaufman [37] and by P. Kranz [40]. In 1978 G. Rodé [52] proved a generalization of the Hahn-Banach Theorem, which is useful in the theory of functional equations and inequalities. Z. Gajda and Z. Kominek [29] presented another approach, which motivated us to prove the following two theorems dealing with the separation problem for inequalities (15) and (16). To be precise, we indicated the conditions for given two solutions of reverse inequalities can be separated by a solution of the equation. Let us first recall the concept of the σ-continuous lattice.

Definition 13 ([8])**.** Let $\{x_n\}$ be a sequence in a lattice $L = (L; \leqslant)$. We call that $x_n \uparrow x$ $(x \in L)$ if and only if $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots, \bigvee x_n$ exists and $\bigvee x_n = x$. In this case we can also write $x = \lim_{n\to\infty} x_n$.
A lattice L is said to be σ-continuous if $x_n \uparrow x$ implies $x_n \wedge y \uparrow x \wedge y$ (or equivalently, $x_n \downarrow x$ implies $x_n \vee y \downarrow x \vee y$) for every $y \in L$. If L is σ-continuous, then for the sequences $\{x_n\}$ and $\{y_n\}$ in L such that $x_n \uparrow x$, $y_n \uparrow y$, we have $x_n \wedge y_n \uparrow x \wedge y$ (or equivalently, $x_n \downarrow x$ and $y_n \downarrow y$ implies $x_n \vee y_n \downarrow x \vee y$).

Theorem 29 ([R13], Theorem 3.1)**.** *Let us be given a σ-continuous lattice L and a commutative semigroup $(S, *)$ which has no elements of finite order, i.e. if $x \in S$, then there is no number $n \geqslant 2$ for which $x^n = x$. Further, let $f, g\colon S \to L$ be functionals for which $g(x * y) \geqslant g(x) \vee g(y)$ and $f(x * y) \leqslant f(x) \vee f(y)$ for all $x, y \in S$. Suppose that $g(x) \leqslant f(x)$ and $\lim_{n\to\infty} g(x^{2^n}) = \lim_{n\to\infty} f(x^{2^n})$ for every $x \in S$. Then there is a functional $a\colon G \to L$ such that*

- ***(i)*** *$g(x) \leqslant a(x) \leqslant f(x)$ for all $x \in S$,*
- ***(ii)*** *$a(x * y) = a(x) \vee a(y)$ for all $x, y \in S$.*

Moreover, the functional $a\colon S \to L$ which meets conditions **(i)** *and* **(ii)** *is unique.*

Theorem 30 ([R13], Theorem 3.2)**.** *Let $(S, *)$ be an Abelian group and $(L, \leqslant)$ a lattice. If mappings $g\colon S \to L$ and $f\colon G \to L$ fulfillinequalities* (15) *and* (16)*, respectively, then*

- ***(i)*** *$g(x) = g(e)$ for every $x \in S$;*
- ***(ii)*** *$f(e) \leqslant f(x) \vee f(x^{-1})$ for every $x \in S$. Moreover, given any $x \in S$, if $f(x) = f(x^{-1})$, then $f(e) \leqslant f(x)$;*
- ***(iii)*** *$f(e) \geqslant f(x) \vee f(x^{-1})$ for all $x \in S$ if and only if $f(x) = f(e)$ for all $x \in S$.*

Furthermore, suppose that $g(x) \leqslant f(x)$ and $f(x) = f(x^{-1})$ for every $x \in S$. Then the functionals f and g can be separated by a constant function, i.e. there exists $\beta \in L$ such that $g(x) \leqslant \beta \leqslant f(x)$ for all $x \in S$.

The next main result in [R13] states that the following functional equation

$$T(x^2) = T(x), \quad x \in S$$

possesses some stability behaviour for mappings defined on a commutative semigroup and taking values in a Banach lattice.

Theorem 31 ([R13], Theorem 3.4)**.** *Let $(S, *)$ be a commutative semigroup and $\mathcal{B}$ be a Banach lattice, $F\colon S \to \mathcal{B}$ and $\Psi\colon S \times S \to [0, +\infty)$ satisfy*

$$\|F(x * y) - F(x) \vee F(y)\| \leqslant \Psi(x, y), \quad x, y \in S. \tag{17}$$

If $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers converging to zero and $\Phi\colon S \to [0, +\infty)$ defined by

$$\Phi(x) = \Psi(x, x), \quad x, y \in S$$

satisfies

$$\lim_{n\to+\infty} \alpha_n \sum_{k=0}^{n-1} \Phi(x^{2^k}) = 0, \quad x \in S, \tag{18}$$

then for every $x \in S$ the sequence $(\alpha_n F(x^{2^n}))_{n\in\mathbb{N}}$ converges to zero in $\mathcal{B}$.

Conversely, if there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of positive real numbers such that for every $x \in S$ the sequence $(\alpha_n F(x^{2^n}))_{n\in\mathbb{N}}$ converges to some $T(x)$ and Ψ satisfies

$$\lim_{n\to+\infty} \alpha_n \Psi(x^{2^n}, y^{2^n}) = 0 \quad x, y \in S, \tag{19}$$

then $T\colon S \to \mathcal{B}$ is a solution of equation (14)*.*

It is worth emphasizing here that in the proof of Theorem 31 no completeness of $\mathcal{B}$ was used. This is a substantial difference between our approach and a vast majority of other stability results, where completeness of the target space is essential. Note however, that in the second part of Theorem 31 this is partially hidden in the assumptions, because we assume that a certain sequence is convergent.

The last group of published papers [R20]-[R22] includes results of collaboration with S. Milles and L. Zedam on intuitionistic fuzzy complete lattices.

Despite of the name, Atanassov intuitionistic fuzzy set (introduced in 1983 as one of the generalizations of the Zadeh fuzzy set [66]) does not have much in common with intuitionism in mathematics and logic (see the discussion on the terminology of intuitionistic fuzzy sets in [25]). To be precise,

Definition 14 ([7])**.** Let $\mathcal{X}$ be a nonempty set. An intuitionistic fuzzy set A on $\mathcal{X}$ is defined as ordered triple $A =^{def} \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{X}\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ represent, respectively, the membership degree and the non-membership degree of the element x in the intuitionistic fuzzy set A for each $x \in X$ with the condition

$$0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1 \text{ for each } x \in \mathcal{X}.$$

The class of all intuitionistic fuzzy sets on $\mathcal{X}$ is denoted by $IFS(\mathcal{X})$.
Moreover, the support of A is the crisp subset of X given by

$$Supp(A) = \{x \in \mathcal{X} : \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}.$$

A weak level set (α, β) (known as (α, β)-cut) of an intuitionistic fuzzy set A is defined as $A_{\alpha,\beta} = \{x \in \mathcal{X} : \mu_A(x) \geqslant \alpha \text{ oraz } \nu_A(x) \leqslant \beta\}$, where $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leqslant 1$.
$C \in IFS(\mathcal{X})$ is called an intuitionistic fuzzy chain if $Supp(C)$ is a crisp chain on $\mathcal{X}$.

The idea of an intuitionistic fuzzy complete lattice, i.e. where $\mathcal{X}$ is a complete lattice, was inspired by the concept of the intuitionistic fuzzy lattice proposed by K.V. Thomas and L.S. Nair in [62].

Definition 15 ([R20], Definition 6)**.** Let L be a complete lattice and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in L\}$ be a IFS of L. Then A is called an intuitionistic fuzzy complete lattice if for any intuitionistic fuzzy set $B \in IFS(L)$ the following conditions are satisfied:

(i) $\mu_A(\sqcup Supp(B)) \geqslant \inf \mu_A(Supp(B)) = \inf_{x \in Supp(B)} \mu_A(x)$,

(ii) $\mu_A(\sqcap Supp(B)) \geqslant \inf \mu_A(Supp(B))$,

(iii) $\nu_A(\vee Supp(B)) \leqslant \sup \nu_A(Supp(B)) = \sup_{x \in Supp(B)} \nu_A(x)$,

(iv) $\nu_A(\wedge Supp(B)) \leqslant \sup \nu_A(Supp(B))$.

Based on the above definition we have made some characterizations of these lattices, in particular, we established criteria for completeness of an intuitionistic fuzzy lattice in terms of a crisp complete lattice, as well as in terms of an intuitionistic fuzzy chain.

Theorem 32 ([R20], Theorem 2)**.** *Let L be a complete lattice and $A \in IFS(L)$. Then*

(1) A is an intuitionistic fuzzy complete lattice if and only if A fullfil conditions (i) and (iii) of Definition 15 for each intuitionistic fuzzy set $B \in IFS(L)$.

(2) A is an intuitionistic fuzzy complete lattice if and only if A fullfil conditions (ii) and (iv) of Definition 15 for each intuitionistic fuzzy set $B \in IFS(L)$.

Theorem 33 ([R20], Theorem 3)**.** *Let L be a complete lattice and $A \in IFS(L)$. Then A is an intuitionistic fuzzy complete lattice if and only if its weak level sets ((α, β)-cuts) are crisp complete lattices.*

Theorem 34 ([R20], Theorem 4)**.** *Let A be an intuitionistic fuzzy lattice. Then the following are equivalent:*

(i) A is an intuitionistic fuzzy complete lattice,

(ii) A is an intuitionistic fuzzy chain-complete,

(iii) every maximal intuitionistic fuzzy chain of A is an intuitionistic fuzzy complete lattice.

In turn, in the paper [R21], and its extended version [R22] covering comprehensive proofs of the main results, we pointed out another important criterion for completeness of intuitionistic fuzzy lattices using fixed points of intuitionistic isotonic mappings.

In 1955, A. Tarski in [61] proved the theorem which states that every isotonic function on a complete lattice has a fixed point. The converse theorem was received A.C. Davis in [18], proving that if every isotonic function defined on a lattice has a fixed point, then this lattice is complete. Both results have established the criterion of completeness of lattices in term fixed points of isotonic mapping. The main result obtained (Theorem 3 in [R21] or Theorem 3 in [R22]), although simple in the description and complicated in the proof, confirms the validity of this criterion also in the case of intuitionistic fuzzy complete lattices.

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Eva Rak