Summary of professional accomplishments

1. Name and surname: Radosław Czaja
2. Scientific degrees

1. Ph.D. in Mathematics, June 29, 2004, Institute of Mathematics of the University of Silesia in Katowice. Title of the Ph.D. Thesis: *Linear and Semilinear Abstract Parabolic Equations*, supervisor: prof. dr hab. Jan Cholewa.

2. Master in Mathematics (Applications of Mathematics), June 2, 2000, Faculty of Mathematics, Physics and Chemistry of the University of Silesia in Katowice. Title of the Master Thesis: *Differential Equations with Sectorial Operator*, supervisor: prof. dr hab. Tomasz Dłotko.

3. University appointments

1. Institute of Mathematics, University of Silesia in Katowice,
X 2004 – present time (including two long-term scientific leaves), Adjunct (Associate Professor).

2. Instituto Superior Técnico, Lisbon, Portugal,
IX 2009 – VIII 2014, Investigador Auxiliar.

3. Instituto Superior Técnico, Lisbon, Portugal,
IX 2005 – VIII 2007, Postdoctoral Fellow.

4. Institute of Mathematics, University of Silesia in Katowice,
X 2001 – IX 2004, Assistant Professor (half-time).

4. Indication of the achievement according to Article 16 Paragraph 2 of the Act of March 14, 2003 on scientific degrees and scientific title and on degrees and title in the field of art (Dz. U. 2016 r. poz. 882 ze zm. w Dz. U. z 2016 r. poz. 1311)

The indicated scientific achievement consists of a series of five publications entitled:

Aspects of asymptotics of semigroups and evolution processes.

4a. List of publications making part of the indicated scientific achievement

[1] Radosław Czaja, Messoud Efendiev, Pullback exponential attractors for nonautonomous equations Part I: Semilinear parabolic problems, *Journal of Mathematical Analysis and Applications* 381 (2011), 748–765.

[2] Radosław Czaja, Messoud Efendiev, Pullback exponential attractors for nonautonomous equations Part II: Applications to reaction-diffusion systems, *Journal of Mathematical Analysis and Applications* 381 (2011), 766–780.

[3] Radosław Czaja, Pullback exponential attractors with admissible exponential growth in the past, *Nonlinear Analysis: Theory, Methods & Applications* 104 (2014), 90–108.

[4] Everaldo de Mello Bonotto, Matheus Cheque Bortolan, Alexandre Nolasco de Carvalho, Radosław Czaja, Global attractors for impulsive dynamical systems - a precompact approach, *Journal of Differential Equations* 259 (2015), 2602–2625.

[5] Radosław Czaja, Carlos Rocha, Transversality in scalar reaction-diffusion equations on a circle, *Journal of Differential Equations* 245 (2008), 692–721.

4b. Description of the above-mentioned publications and obtained results

Introduction

Physical and biological processes evolving in time are modeled most often by the use of ordinary and partial differential equations and systems. If the initial or initial boundary value problem for a differential equation is globally well-posed, i.e., solutions exist, are unique, can be extended for all times and depend continuously on the initial conditions, then, in the case of autonomous equations, such a problem generates a dynamical system or a semigroup $\{T(t)\colon t \geqslant 0\}$ on a given metric space (V, d) called the phase space.

Among many questions concerning semigroups, one of the most important is the study of behavior of trajectories $t \mapsto T(t)u_0$ in time, where we are not interested in transient behavior, but in asymptotic one as $t \to \infty$. Particularly interesting are physical systems in which dissipation of energy takes place. They are described by dissipative semigroups for which there exists a bounded set B_0, which attracts each bounded subset B of the phase space with respect to the Hausdorff semidistance

$$\mathrm{dist}_V(T(t)B, B_0) = \sup_{x\in B} \inf_{y\in B_0} d(T(t)x, y) \to 0 \text{ as } t \to \infty.$$

The study of the asymptotics of dissipative semigroups on subsets of infinite-dimensional Banach spaces generated by autonomous partial differential equations can be reduced to the description of a global attractor $\mathcal{A}$ being a compact invariant set,

$$T(t)\mathcal{A} = \mathcal{A},\ t \geqslant 0,$$

attracting all bounded subsets of the phase space. Indeed, each trajectory $t \to T(t)u_0$ possesses after a sufficiently long time its "shadow" in the form of a trajectory in the global attractor ([RO, Proposition 10.14]). Therefore the following topics are so important: existence of the global attractor, its characterization, structure and geometry, dynamics on the attractor or its structural stability under the influence of a perturbation of the equation leading to it. These subjects have been investigated all over the world in the span of many years and there exists an extensive literature devoted to them (among others [HE1], [HA1], [LA], [B-V], [TE], [C-D], [RO], [S-Y]).

The global attractor for a semigroup is a uniquely determined object and frequently has a finite (fractal) dimension, but its attraction can be arbitrarily slow or the object itself may not be visible in numerical simulations (cf. [E-Y-Y]). The need to overcome these drawbacks motivated the appearance of the notion of an exponential attractor. The exponential attractor $\mathcal{M}$ for a semigroup is a compact, positively invariant set,

$$T(t)\mathcal{M} \subset \mathcal{M},\ t \geqslant 0,$$

with finite fractal dimension

$$\dim_f^V(\mathcal{M}) = \limsup_{\varepsilon\to 0} \log_{\frac{1}{\varepsilon}} N_\varepsilon^V(\mathcal{M}) < \infty,$$

where $N_\varepsilon^V(\mathcal{M})$ denotes the smallest number of balls of radius ε in V necessary to cover $\mathcal{M}$, and exponentially attracting each bounded subset B of the phase space at uniform rate $\omega > 0$

$$\lim_{t\to\infty} e^{\omega t} \operatorname{dist}_V(T(t)B, \mathcal{M}) = 0.$$

Although such an object is not uniquely determined, it still contains the global attractor $\mathcal{A}$. Moreover, its existence also implies the finite fractal dimension of the global attractor. The first constructions of an exponential attractor come from the monograph [E-F-N-T] and papers [D-N], [E-M-Z].

In recent years more and more attention has been devoted to more general, nonautonomous ordinary and partial differential equations and systems. In this case the counterpart of a semigroup is an evolution process $\{U(t,s)\colon t \geqslant s\}$ on a phase space V. However, there is no unique counterpart of the global attractor. Different approaches usually lead to different notions describing asymptotic behavior of evolution processes such as uniform attractors, pullback global attractors, forward global attractors (cf. [CH], [C-V], [C-L-R], [K-R]). According to the monograph [C-L-R], the most universal of them seems to be the notion of a pullback global attractor, that is, a family of compact sets $\{\mathcal{A}(t)\colon t \in \mathbb{R}\}$, which is invariant under the process,

$$U(t,s)\mathcal{A}(s) = \mathcal{A}(t),\ t \geqslant s,$$

attracts in the pullback sense each bounded subset B of the phase space

$$\operatorname{dist}_V(U(t,s)B, \mathcal{A}(t)) \to 0 \text{ as } s \to -\infty \text{ for every } t \in \mathbb{R},$$

and is minimal in the sense of inclusion among families of closed sets attracting in the pullback sense all bounded subsets (see [C-L-R], [C-Ł-R], [C-C-L-R]).

Pullback exponential attractors for evolution processes

In 2009 there still did not exist general constructions of the counterpart of an exponential attractor for evolution processes, except for the discrete counterpart and a particular case for reaction-diffusion equations from the paper [E-Z-M]. A construction of such an object is contained in my paper [1], written jointly with Messoud Efendiev, which is a part of my scientific achievement. The article [1] was one of the first three papers, independently of [L-M-R] and [E-Y-Y], devoted to general conditions for the existence of a family of compact sets $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ in a Banach space V, which is positively invariant under the process $\{U(t,s)\colon t \geqslant s\}$, i.e.,

$$U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t),\ t \geqslant s,$$

has uniform bound with respect to $t \in \mathbb{R}$ in V of the fractal dimension and exponentially attracts in the pullback sense all bounded subsets of the space V, i.e., there exists $\omega > 0$ such that for every bounded set $B \subset V$ we have

$$\lim_{s\to\infty} e^{\omega s} \operatorname{dist}_V(U(t,t-s)B, \mathcal{M}(t)) = 0 \text{ for every } t \in \mathbb{R}. \tag{1}$$

The family $\{\mathcal{M}(t)\colon t\in\mathbb{R}\}$ is called a pullback exponential attractor.

To present precisely the results of [1], consider an evolution process $\{U(t,s)\colon t\geqslant s\}$ on a Banach space V with the norm $\|\cdot\|_V$. Thus, we have

$$U(t,s)U(s,r)=U(t,r),\ t\geqslant s\geqslant r,\ U(t,t)=Id,\ t\in\mathbb{R}.$$

Because we are interested in the past, we distinguish $t_0\leqslant\infty$ and the set $\mathcal{T}=\{t\in\mathbb{R}\colon t\leqslant t_0\}$. Our result from 2011 deals with processes for which there exists a bounded set $B_0\subset V$ absorbing all bounded sets $B\subset V$ in the pullback sense uniformly w.r.t. $t\in\mathcal{T}$, i.e.,

$$\exists_{T_B>0}\forall_{s\geqslant T_B}\bigcup_{t\in\mathcal{T}}U(t,t-s)B\subset B_0. \tag{2}$$

The next assumption, central for the proof, is the so called smoothing property, already used earlier e.g. in [M-P] to show the finiteness of the fractal dimension of a set. Namely, we assume that the space V is compactly embedded into some auxiliary normed space W with norm $\|\cdot\|_W$ and the process on B_0 satisfies, uniformly w.r.t. $t\in\mathcal{T}$, the following condition with $\kappa>0$

$$\sup_{t\in\mathcal{T}}\|U(t,t-T_{B_0})u_1-U(t,t-T_{B_0})u_2\|_V\leqslant\kappa\|u_1-u_2\|_W,\ u_1,u_2\in B_0, \tag{3}$$

where $T_{B_0}>0$ is the time of absorption of B_0 from (2). Furthermore, we also assume that the process is Hölder continuous w.r.t. the initial time for times from $[T_{B_0},2T_{B_0}]$ and is Hölder continuous also w.r.t. the time shift, i.e., there exist $0<\xi_1,\xi_2\leqslant 1$ and constants $c_1,c_2>0$ such that

$$\sup_{t\in\mathcal{T}}\|U(t,t-t_1)u-U(t,t-t_2)u\|_W\leqslant c_1|t_1-t_2|^{\xi_1},\ t_1,t_2\in[T_{B_0},2T_{B_0}],\ u\in B_0, \tag{4}$$

$$\sup_{t\in\mathcal{T}}\|U(t,t-T_{B_0})u-U(t-t_1,t-t_1-T_{B_0})u\|_W\leqslant c_2t_1^{\xi_2},\ t_1\in[0,T_{B_0}],\ u\in B_0. \tag{5}$$

Of course, for autonomous evolution processes, coming from a semigroup, the above condition (5) is satisfied trivially. In [1, Theorem 2.1] we show that then there exists a family $\{\mathcal{M}(t)\colon t\in\mathcal{T}\}$ of nonempty subsets of B_0, which are precompact (i.e., their closure is compact) in V, which is positively invariant under the process, i.e.,

$$U(t,s)\mathcal{M}(s)\subset\mathcal{M}(t),\ t\geqslant s,\ t\in\mathcal{T}.$$

The mentioned family has a uniform bound w.r.t. $t\in\mathcal{T}$ of the fractal dimension in V expressed by the constants appearing in the assumptions stated above and a parameter $\nu\in(0,\frac{1}{2})$, i.e.,

$$\sup_{t\in\mathcal{T}}\dim_f^V(\mathcal{M}(t))\leqslant\max\{\xi_1^{-1},\xi_2^{-1}\}(1+\log_{\frac{1}{2\nu}}(1+\mu\kappa))+\log_{\frac{1}{2\nu}}N^W_{\frac{\nu}{\kappa}}(B^V(0,1)),$$

where $\mu>0$ is a constant from the embedding of V into W

$$\|u\|_W\leqslant\mu\|u\|_V,\ u\in V,$$

whereas $N^W_{\frac{\nu}{\kappa}}(B^V(0,1))$ denotes the smallest number of balls in W with radius $\frac{\nu}{\kappa}$ necessary to cover a unit ball in V. Moreover, this family exponentially attracts in the pullback sense all bounded subsets of V. More precisely, there exists $\chi > 0$ such that for each bounded $B \subset V$ there is $c_B > 0$ such that

$$\sup_{t\in\mathcal{T}} \operatorname{dist}_V(U(t,t-s)B, \mathcal{M}(t)) \leqslant c_B e^{-\chi s}, \; s \geqslant T_B + 2.$$

If we require additionally that the mapping

$$\operatorname{cl}_V B_0 \ni u \mapsto U(t,s)u \in V \tag{6}$$

is continuous for $t \geqslant s$, $t \in \mathcal{T}$, then, taking closures in V of the above sets $\mathcal{M}(t)$, we can consider that the sets forming the family are compact and contained in $\operatorname{cl}_V B_0$. Let us point out that the family from the construction in [1, Theorem 2.1] is not uniquely determined, because of its dependence on the parameter ν, for example.

According to [1, Proposition 2.3], under the assumptions (2)-(6), there also exists a family $\{\mathcal{A}(t)\colon t \in \mathcal{T}\}$ of compact subsets of V, positively invariant under the process,

$$U(t,s)\mathcal{A}(s) = \mathcal{A}(t), \; t \geqslant s, \; t \in \mathcal{T},$$

which attracts in the pullback sense all bounded sets $B \subset V$ for every $t \in \mathcal{T}$

$$\lim_{s\to\infty} \operatorname{dist}_V(U(t,t-s)B, \mathcal{A}(t)) = 0$$

and if $\{\tilde{\mathcal{A}}(t)\colon t \in \mathcal{T}\}$ is a family of closed sets in V attracting in the pullback sense all bounded sets in V for every $t \in \mathcal{T}$, then $\mathcal{A}(t) \subset \tilde{\mathcal{A}}(t)$, $t \in \mathcal{T}$. The family is given by the formula

$$\mathcal{A}(t) = \operatorname{cl}_V \bigcup_{B\subset V, \text{ bounded}} \omega(B,t), \; t \in \mathcal{T},$$

where $\omega(B,t)$ is the pullback $\omega-$limit set of a subset B at time t

$$\omega(B,t) = \bigcap_{\tau\geqslant 0} \operatorname{cl}_V \bigcup_{s\geqslant\tau} U(t,t-s)B.$$

In particular, if $t_0 = \infty$ we obtain the existence of the pullback global attractor with uniformly bounded fractal dimension, contained in a pullback exponential attractor, which is a subset of $\operatorname{cl}_V B_0$. Moreover, the attraction of bounded sets by the family $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ is uniform, i.e., there exists $\omega > 0$ such that for every bounded set $B \subset V$

$$\lim_{s\to\infty} e^{\omega s} \sup_{t\in\mathbb{R}} \operatorname{dist}_V(U(t,t-s)B, \mathcal{M}(t)) = 0$$

holds, which in turn is equivalent to the forward exponential uniform attraction, i.e.,

$$\lim_{s\to\infty} e^{\omega s} \sup_{t\in\mathbb{R}} \operatorname{dist}_V(U(t+s,t)B, \mathcal{M}(t+s)) = 0.$$

In [1, Corollary 2.4] we observe that to obtain the existence of a pullback exponential attractor we do not need to assume that $t_0 = \infty$. In the case of $t_0 < \infty$ it is enough to assume the Lipschitz continuity of the process

$$\forall_{t>0} \exists_{k(t)>0} \forall_{u_1,u_2\in B_0} \left\| U(t+t_0,t_0)u_1 - U(t+t_0,t_0)u_2 \right\|_V \leqslant k(t) \left\| u_1 - u_2 \right\|_V \tag{7}$$

and define the missing sets of the family by $\mathcal{M}(t) = U(t, t_0)\mathcal{M}(t_0)$, $t \geqslant t_0$. Although we then lose the uniform character of the exponential attraction, we still know that (1) holds. Of course, the existence of the pullback global attractor contained in the pullback exponential attractor is also guaranteed in this case (see [1, Proposition 2.5]).

In the paper [1] we formulated conditions for nonautonomous semilinear parabolic equations to generate an evolution process satisfying assumptions (2)–(7). Consider the following abstract Cauchy problem

$$\begin{cases} u_t + Au = F(t, u), \ t > s, \\ u(s) = u_0, \end{cases} \tag{8}$$

where A is a positive sectorial operator (cf. [HE1], [C-D], [6]) in a Banach space X with a compact resolvent. By $X^\gamma = D(A^\gamma)$ we denote the fractional power spaces corresponding to the operator A. We fix $\alpha \in [0, 1)$ and assume that the nonlinearity $F \colon \mathbb{R} \times X^\alpha \to X$ is Hölder continuous with respect to time and Lipschitz continuous on bounded subsets of X^α. More precisely, for every bounded set $B \subset X^\alpha$ there exists $0 < \theta = \theta(B) < 1$ such that for any $T_1, T_2 \in \mathbb{R}, T_1 < T_2$ there is a Lipschitz constant $L = L(T_2 - T_1, B) > 0$ such that

$$\|F(t_1, u_1) - F(t_2, u_2)\|_X \leqslant L(|t_1 - t_2|^\theta + \|u_1 - u_2\|_{X^\alpha}), \ t_1, t_2 \in [T_1, T_2], \ u_1, u_2 \in B. \tag{9}$$

Under the assumption (9) for any initial time $s \in \mathbb{R}$ and any initial condition $u_0 \in X^\alpha$ there exists a unique X^α solution of the problem (8), i.e.,

$$u \in C([s, t_{max}), X^\alpha) \cap C((s, t_{max}), X^1) \cap C^1((s, t_{max}), X)$$

satisfying the differential equation from (8) in X and defined on the maximal interval of existence $[s, t_{max})$ (cf. [HE1], [C-D]).

We distinguish the set $\mathcal{T} = \{t \in \mathbb{R} \colon t \leqslant t_0\}$ for some $t_0 \leqslant \infty$ and assume that for some $M > 0$

$$\sup_{t \in \mathcal{T}} \|F(t, 0)\|_X \leqslant M. \tag{10}$$

To prove that local solutions can be extended to the whole half-line and obtain the existence of a bounded absorbing set in X^α, we check in applications an appropriate a priori estimate. In consequence, we assume that

$$\text{every local solution can be extended to the global one, i.e., } t_{max} = \infty, \tag{11}$$

there exists a constant $a > 0$ and a nondecreasing function $Q \colon [0, \infty) \to [0, \infty)$ (both independent of s) such that

$$\|u(t)\|_{X^\alpha} \leqslant Q(\|u_0\|_{X^\alpha}) e^{-a(t-s)} + R_0, \ s \leqslant t, \ t \in \mathcal{T}, \tag{12}$$

holds with a constant $R_0 = R_0(t_0) > 0$ independent of s, t and u_0 and (in the case of $t_0 < \infty$) for every $T > 0$ there exist $R_{T,s} > 0$ and a nondecreasing function $\widetilde{Q}_{T,s} \colon [0, \infty) \to [0, \infty)$ such that

$$\|u(t)\|_{X^\alpha} \leqslant \widetilde{Q}_{T,s}(\|u_0\|_{X^\alpha}) + R_{T,s}, \ t \in [s, s + T]. \tag{13}$$

The assumptions (11)–(13) can be simplified by replacing them with a stronger a priori condition, which guarantees dissipativity in X^α. Namely, let

$$\|u(t)\|_{X^\alpha} \leqslant Q(\|u_0\|_{X^\alpha}) e^{-a(t-s)} + R(t), \ t \in [s, t_{max}), \tag{14}$$

where $a > 0$, $Q\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function and $R\colon \mathbb{R} \to [0,\infty)$ is a continuous function bounded on $\mathcal{T}$.

Under the assumptions (9)–(13) the X^α solutions of the problem (8) exist globally in time and generate an evolution process $\{U(t,s)\colon t \geqslant s\}$ on X^α, which satisfies the assumptions (2)–(7) with $V = X^\beta$ and $W = X^\alpha$ for $\beta \in (\alpha, 1)$.

Theorem 1 ([1, Theorem 3.6]). *Under the above assumptions for $\beta \in (\alpha,1)$ there exists a family $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ of nonempty compact subsets of X^β, positively invariant under the process $\{U(t,s)\colon t \geqslant s\}$, which has a uniform bound w.r.t. $t \in \mathbb{R}$ of the fractal dimension in X^β and exponentially attracts in the pullback sense all bounded subsets of X^β. Additionally, if $t_0 = \infty$, then the exponential attraction is uniform w.r.t. $t \in \mathbb{R}$. Moreover, the pullback exponential attractor $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ contains the pullback global attractor $\{\mathcal{A}(t)\colon t \in \mathbb{R}\}$ in the space X^β.*

The paper [2], which makes part of the scientific achievement, is a natural illustration of the topics discussed in [1] and contains applications of the abstract theory to nonautonomous reaction-diffusion equations and systems.

In the main part of the paper [2] we consider, following [E-Z], the nonautonomous reaction-diffusion system of equations

$$\begin{cases} u_t + Au = f(u) + g(t),\ t > s,\ x \in \Omega, \\ u(s,x) = u_0(x),\ x \in \Omega, \quad u(t,x) = 0,\ t \geqslant s,\ x \in \partial\Omega, \end{cases} \tag{15}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with the boundary $\partial\Omega$ of class $C^{2+\eta}$. Here $u(t,x) = (u_1(t,x),\ldots,u_k(t,x))$ is an unknown function, whereas $f(u) = (f_1(u),\ldots,f_k(u))$ and $g(t,x) = (g_1(t,x),\ldots,g_k(t,x))$ are given functions. We assume that $Au = (A_1u_1,\ldots,A_ku_k)$ is a second order elliptic differential operator, where

$$A_l u_l(x) = \sum_{i,j=1}^{3} \partial_{x_i}(a_{ij}^l(x)\partial_{x_j} u_l(x)),\ x \in \Omega,\ l = 1,\ldots,k,$$

with coefficients $a_{ij}^l = a_{ji}^l$ of class $C^{1+\eta}(\overline{\Omega})$ and satisfying the uniform strong ellipticity condition

$$\exists_{\nu>0}\forall_{l=1,\ldots,k}\forall_{x\in\Omega}\forall_{\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3} \quad -\sum_{i,j=1}^{3} a_{ij}^l(x)\xi_i\xi_j \geqslant \nu\,|\xi|^2 .$$

Furthermore, we assume that for the nonlinearity $f \in C(\mathbb{R}^k,\mathbb{R}^k)$ there exist constants $p_1,\ldots,p_k \geqslant 0$ and $q_1,\ldots,q_k \geqslant 0$ such that f satisfies the growth condition

$$\exists_{c>0}\forall_{u=(u_1,\ldots,u_k),v=(v_1,\ldots,v_k)\in\mathbb{R}^k}\ |f(u)-f(v)|^2 \leqslant c\sum_{l=1}^{k} |u_l - v_l|^2 \left(1 + |u_l|^{p_l} + |v_l|^{p_l}\right) \tag{16}$$

and the anisotropic dissipativity condition

$$\exists_{C>0}\forall_{u=(u_1,\ldots,u_k)\in\mathbb{R}^k}\ \sum_{l=1}^{k} f_l(u)u_l\,|u_l|^{q_l} \leqslant C. \tag{17}$$

As refers to the time-dependent perturbation, we suppose that

$$g\colon \mathbb{R} \to [L^2(\Omega)]^k \text{ is globally Hölder continuous with exponent } \theta \in (0,1] \tag{18}$$

and there exists $t_0 \leqslant \infty$ such that

$$\sup_{t\in\mathcal{T}} \|g(t)\|_{[L^2(\Omega)]^k} < \infty, \tag{19}$$

where we denoted $\mathcal{T} = \{t \in \mathbb{R} \colon t \leqslant t_0\}$ as above.

In particular, if $k = 2$ and for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$f_1(u_1, u_2) = \alpha u_1 + \beta u_1^2 - u_1^3 - \gamma u_2, \quad f_2(u_1, u_2) = \delta u_1 - \varepsilon u_2, \tag{20}$$

then the system (15) is a nonautonomous perturbation of the FitzHugh-Nagumo system modeling transmission of nerve impulses in axons. In this case, both assumptions (16) and (17) are satisfied with $p_1 = 4$, $p_2 = 0$ and $q_1 = q_2 = q$, where $q \geqslant 0$ is arbitrary.

Another particular case of the system (15) is the model of a chemical reaction with the nonlinearity

$$f_1(u_1, u_2) = u_2 - u_1^3, \quad f_2(u_1, u_2) = u_1^3 - u_2. \tag{21}$$

Then the assumptions (16) and (17) are satisfied with $p_1 = 4$, $p_2 = 0$ and $q_1 = 4$, $q_2 = \frac{2}{3}$, whereas the common dissipativity condition ($q_1 = q_2 = 0$) does not hold, since the expression

$$(u_2 - u_1^3)u_1 + (u_1^3 - u_2)u_2 = (u_2 - u_1)(u_1^3 - u_2)$$

attains values arbitrarily large.

We consider (15) as the abstract Cauchy problem (8) in the space $X = [L^2(\Omega)]^k$ with $F(t, u) = f(u) + g(t)$, where A is a sectorial operator in X with the domain $D(A) = [H^2(\Omega) \cap H_0^1(\Omega)]^k$ and has a compact resolvent.

First we check in [2, Proposition 3.2] that $F \colon \mathbb{R} \times X^{\frac{1}{2}} \to X$, where $X^{\frac{1}{2}} = [H_0^1(\Omega)]^k$ is the fractional power space, is well-defined and satisfies the condition (9) provided that $0 \leqslant p_l \leqslant 4$, $l = 1, \ldots, k$. The assumption (10) can be easily verified, since it follows from (19) that

$$\sup_{t\in\mathcal{T}} \|F(t, 0)\|_{[L^2(\Omega)]^k} \leqslant \|f(0)\|_{[L^2(\Omega)]^k} + \sup_{t\in\mathcal{T}} \|g(t)\|_{[L^2(\Omega)]^k} < \infty.$$

The most difficult stage of the study of the system (15) was to prove that under certain conditions on p_l and q_l the assumptions (11)–(13) are satisfied. In succession we showed the a priori estimates of

$$\sum_{l=1}^{k} \|u_l(t)\|_{L^{2+q_l}(\Omega)}^{2+q_l} \text{ and } \int_{t-h}^{t} \sum_{l=1}^{k} \left\|\left|\nabla(|u_l(\tau)|^{\frac{q_l+2}{2}})\right|\right\|_{L^2(\Omega)}^2 d\tau$$

in [2, Proposition 3.5], the a priori estimates of

$$\|u(t)\|_{[L^2(\Omega)]^k}^2 \text{ and } \int_{t-h}^{t} \sum_{l=1}^{k} \||\nabla u_l(\tau)|\|_{L^2(\Omega)}^2 \, d\tau$$

in [2, Proposition 3.7] and finally the a priori estimate of

$$\sum_{l=1}^{k} \||\nabla u_l(t)|\|_{L^2(\Omega)}^2$$

in [2, Proposition 3.8]. They lead to the following result in which we verify the assumptions (11)–(13).

Theorem 2 ([2, Corollary 3.9]). *If $p_l \leqslant q_l \leqslant 4$, $l = 1, \dots, k$ and $u = (u_1, \dots, u_k)$ is an $X^{\frac{1}{2}}$ solution of* (15) *on $[s, t_{max})$, then $t_{max} = \infty$ and for $t \geqslant s$, $t \in \mathcal{T}$ we have*

$$\|u(t)\|_{[H_0^1(\Omega)]^k} \leqslant Q_1\Big(\|u(s)\|_{[H_0^1(\Omega)]^k} \Big) e^{-\frac{\lambda_1 \nu}{8}(t-s)} + Q_2\Big(\sup_{\tau \in (-\infty, t_0+2)} \|g(\tau)\|_{[L^2(\Omega)]^k} \Big),$$

where Q_1, Q_2 are positive nondecreasing functions, and for any $T > 0$ there exist positive nondecreasing functions $\widetilde{Q}_1 = \widetilde{Q}_1(T)$, $\widetilde{Q}_2 = \widetilde{Q}_2(T)$ such that for $s \leqslant t \leqslant s + T$

$$\|u(t)\|_{[H_0^1(\Omega)]^k} \leqslant \widetilde{Q}_1\Big(\|u(s)\|_{[H_0^1(\Omega)]^k} \Big) e^{-\frac{\lambda_1 \nu}{8}(t-s)} + \widetilde{Q}_2\Big(\sup_{\tau \in [s, s+T]} \|g(\tau)\|_{[L^2(\Omega)]^k} \Big)$$

holds, where $\lambda_1 > 0$ is the constant from the Poincaré inequality.

On account of that we can apply Theorem 1. Thus the main result of the paper [2] is the theorem [2, Theorem 3.10] on the existence of a pullback exponential attractor and the pullback global attractor with uniformly bounded fractal dimension in the space $[H_0^{2\beta}(\Omega)]^k$ with $\beta \in (\frac{1}{2}, 1)$ for the problem (15) if the conditions (16) and (17) hold with $0 \leqslant p_l \leqslant q_l \leqslant 4$, $l = 1, \dots, k$, and the nonautonomous perturbation satisfies (18) and (19). In particular, this applies to the perturbation of the FitzHugh-Nagumo system (20) and the nonlinearity (21).

Another application of the theory introduced in [1] is the initial boundary value problem of Dirichlet type for the nonautonomous Chafee-Infante equation of the form

$$\begin{cases} u_t = \triangle_D u + \lambda u - b(t) u^3, \ t > s, \ x \in \Omega, \\ u(s, x) = u_0(x), \ x \in \Omega, \quad u(t, x) = 0, \ t \geqslant s, \ x \in \partial\Omega, \end{cases} \tag{22}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \leqslant 3$, with sufficiently smooth boundary $\partial\Omega$, which was studied earlier among others in the paper [L-S]. In our case we assume that $\lambda \in \mathbb{R}$ and the function b is Hölder continuous on $\mathbb{R}$ with exponent $\theta \in (0, 1]$ and satisfies

$$0 < b(t) \leqslant M, \ t \in \mathbb{R},$$

with some constant $M > 0$. Moreover, we assume that there exist $t_0 \leqslant \infty$ and $m > 0$ such that

$$m \leqslant b(t), \ t \in \mathcal{T} = \{t \in \mathbb{R} \colon t \leqslant t_0\}.$$

We observe that the problem (22) can be considered as the abstract Cauchy problem (8) with $A = -\triangle_D$ in $X = L^2(\Omega)$ with the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $F \colon \mathbb{R} \times X^{\frac{1}{2}} \to X$ given by $F(t, u) = \lambda u - b(t) u^3$, where $X^{\frac{1}{2}} = H_0^1(\Omega)$, which satisfies the assumptions (9) and (10). To verify the condition (14), we show the a priori estimate in the space $H_0^1(\Omega)$. Specifically, we get

$$\|u(t)\|_{H_0^1(\Omega)} \leqslant \sqrt{1 + 2|\lambda| + \lambda_1} \, \|u(s)\|_{H_0^1(\Omega)} \, e^{-\frac{\lambda_1}{2}(t-s)} + R(t),$$

where

$$R(t) = R_0 \left(\lambda_1 m \int_{-\infty}^{t} \frac{e^{-\lambda_1 (t-\tau)}}{b(\tau)} d\tau \right)^{\frac{1}{2}}, \ t \in \mathbb{R},$$

and

$$R_0 = \sqrt{\frac{(1 + 2|\lambda| + \lambda_1) \lambda^2 |\Omega|}{2 \lambda_1 m}},$$

with $\lambda_1 > 0$ being the first eigenvalue of the Laplace operator considered here.

Therefore, we can apply Theorem 1 and obtain for (22) the existence of a pullback exponential attractor and the pullback global attractor with uniformly bounded fractal dimension in the space $H_0^{2\beta}(\Omega)$ for $\beta \in (\frac{1}{2}, 1)$, which is stated in [2, Corollary 2.1].

I presented the results of the previously described paper [1], for example, during the ICMC Summer Meeting on Differential Equations Chapter 2011 in São Carlos in Brazil in 2011. It was there that our construction aroused interest of Alexandre Nolasco de Carvalho and Stefanie Sonner, who simplified and generalized it in 2013 in the paper [C-S1]. First, they let the process $\{U(t,s)\colon t \geqslant s\}$ on V be asymptotically compact as the sum $U = S + C$, where the family of operators S has smoothing property with respect to the space V and an auxiliary space W with V compactly embedded into W, whereas C is a family of contractions in the space V. The other important aspect was to allow the dependence of the absorbing set B_0 on time admitting the unboundedness of the pullback exponential attractor also in the past. However, the mentioned dependence on time of the absorbing family $\{B(t)\colon t \in \mathbb{R}\}$ could not be exponential in the paper [C-S1], since the sets $B(t)$ could grow in the past only subexponentially. Removal of this assumption constitutes the main result of my paper [3], which is a part of the scientific achievement.

In [3] I assume that

($\mathcal{A}_1$) there exists a family of nonempty closed and bounded subsets $B(t)$, $t \in \mathbb{R}$, of a Banach space V, which is positively invariant under the evolution process $\{U(t,s)\colon t \geqslant s\}$ on V, i.e., $U(t,s)B(s) \subset B(t)$, $t \geqslant s$,

($\mathcal{A}_2$) there exist $t_0 \in \mathbb{R}$, $\gamma_0 \geqslant 0$ and $M > 0$ such that

$$\mathrm{diam}_V(B(t)) < Me^{-\gamma_0 t},\ t \leqslant t_0,$$

($\mathcal{A}_3$) in the past the family $\{B(t)\colon t \in \mathbb{R}\}$ absorbs in the pullback sense all bounded subsets of V, i.e., for every bounded set D in V and $t \leqslant t_0$ there exists $T_{D,t} \geqslant 0$ such that

$$U(t, t-r)D \subset B(t),\ r \geqslant T_{D,t},$$

and, additionally, the function $(-\infty, t_0] \ni t \mapsto T_{D,t} \in [0, \infty)$ is nondecreasing for each such D, so in fact we have for every bounded D in V and $t \leqslant t_0$

$$U(s, s-r)D \subset B(s),\ s \leqslant t,\ r \geqslant T_{D,t}.$$

Observe that ($\mathcal{A}_2$) implies that for every $\gamma > \gamma_0$

$$\mathrm{diam}_V(B(t))e^{\gamma t} \to 0 \text{ as } t \to -\infty,$$

which generalizes the assumption used in [C-S1, Definition 3.1]. In particular, the assumptions stated above admit an exponential growth in the past of the sets forming the pullback absorbing family.

Next, I assume that the family of operators $\{U(t,s)\colon t_0 \geqslant t \geqslant s\}$ can be decomposed as a sum

$$U(t,s) = C(t,s) + S(t,s),$$

where $\{C(t,s)\colon t_0 \geqslant t \geqslant s\}$ and $\{S(t,s)\colon t_0 \geqslant t \geqslant s\}$ are families of operators satisfying the following properties:

($\mathcal{H}_1$) there exists $\tilde{t} > 0$ such that $C(t, t-\tilde{t})$ are contractions on the absorbing family with contraction constant independent of time, i.e.,

$$\left\| C(t, t-\tilde{t})u - C(t, t-\tilde{t})v \right\|_V \leqslant \lambda \left\| u - v \right\|_V, \ t \leqslant t_0, \ u, v \in B(t-\tilde{t}),$$

where $0 \leqslant \lambda < \frac{1}{2}e^{-\gamma_0 \tilde{t}}$ with $\gamma_0 \geqslant 0$ taken from the assumption ($\mathcal{A}_2$),

($\mathcal{H}_2$) there exists an auxiliary normed space $(W, \|\cdot\|_W)$ such that V is compactly embedded into W and $\mu > 0$ satisfies

$$\|u\|_W \leqslant \mu \|u\|_V, \ u \in V,$$

whereas the operators $S(t, t-\tilde{t})$ satisfy the smoothing property with a constant $\kappa > 0$, that is

$$\left\| S(t, t-\tilde{t})u - S(t, t-\tilde{t})v \right\|_V \leqslant \kappa \left\| u - v \right\|_W, \ t \leqslant t_0, \ u, v \in B(t-\tilde{t}).$$

Finally, I also assume that

($\mathcal{H}_3$) the process is Lipschitz continuous on the absorbing family, i.e., for every $t \in \mathbb{R}$ and $s \in [t, t+\tilde{t}]$ there exists a constant $L_{t,s} > 0$ such that

$$\|U(s,t)u - U(s,t)v\|_V \leqslant L_{t,s} \|u - v\|_V, \ u, v \in B(t).$$

The assumption ($\mathcal{H}_3$) in fact implies that for any $s \geqslant t$ there is a constant $L_{t,s} > 0$ such that

$$\|U(s,t)u - U(s,t)v\|_V \leqslant L_{t,s} \|u - v\|_V, \ u, v \in B(t).$$

Observe that the assumptions ($\mathcal{A}_1$) and ($\mathcal{H}_3$) hold for every $t \in \mathbb{R}$, while the rest of the assumptions is satisfied only in the past, i.e., for $t \leqslant t_0$.

The main result of the paper [3] is the theorem on the existence of a pullback exponential attractor under the above-mentioned assumptions, which admit in the past an exponential growth of the pullback absorbing family.

Theorem 3 ([3, Theorem 2.2])**.** *If the process $\{U(t,s)\colon t \geqslant s\}$ on a Banach space V satisfies ($\mathcal{A}_1$)-($\mathcal{A}_3$) and ($\mathcal{H}_1$)-($\mathcal{H}_3$), then for every $\nu \in (0, \frac{1}{2}e^{-\gamma_0 \tilde{t}} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}(t) = \mathcal{M}^\nu(t)\colon t \in \mathbb{R}\}$ in V with the following properties:*

(a) $\mathcal{M}(t)$ is a nonempty compact subset of $B(t)$ for $t \in \mathbb{R}$,

(b) $U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t)$, $t \geqslant s$,

(c) the fractal dimension of the set $\mathcal{M}(t)$ is uniformly bounded w.r.t. $t \in \mathbb{R}$, namely

$$\sup_{t \in \mathbb{R}} \dim_f^V(\mathcal{M}(t)) \leqslant \frac{-\ln N^W_{\frac{\nu}{\kappa}}(B^V_1(0))}{\ln\left(2(\nu+\lambda)\right) + \gamma_0 \tilde{t}},$$

where $N^W_{\frac{\nu}{\kappa}}(B^V_1(0))$ denotes the smallest number of balls in W with radius $\frac{\nu}{\kappa}$ and centers in $B^V_1(0)$ necessary to cover the unit ball $B^V_1(0)$ in V,

(d) *for any* $t \in \mathbb{R}$ *there exists* $c_t > 0$ *such that for any* $s \geqslant \max\{t - t_0, 0\} + 2\tilde{t}$

$$\operatorname{dist}_V(U(t, t-s)B(t-s), \mathcal{M}(t)) \leqslant c_t e^{-\omega_0 s},$$

where $\omega_0 = -\frac{1}{\tilde{t}}\left(\ln\left(2(\nu+\lambda)\right) + \gamma_0 \tilde{t}\right) > 0$,

(e) *for any* $0 < \omega < \omega_0$ *and any bounded set* $D \subset V$ *we have*

$$\lim_{s \to \infty} e^{\omega s} \operatorname{dist}_V(U(t, t-s)D, \mathcal{M}(t)) = 0, \ t \in \mathbb{R}.$$

In [3, Corollary 2.6] I formulated a more general condition, which can substitute the smoothing property $(\mathcal{H}_2)$ in Theorem 3. Namely,

(H_2) there is $N = N_\nu \in \mathbb{N}$ such that for any $t \leqslant t_0$, any $R > 0$ and any $u \in B(t-\tilde{t})$ there exist $v_1, \ldots, v_N \in V$ such that

$$S(t, t-\tilde{t})(B(t-\tilde{t}) \cap B^V_R(u)) \subset \bigcup_{i=1}^{N} B^V_{\nu R}(v_i).$$

Of course the existence of a pullback exponential attractor implies the existence of the pullback global attractor with uniformly bounded fractal dimension. More precisely, on account of [3, Corollary 2.8] we have

$$\sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{A}(t)) \leqslant \sup_{t\in\mathbb{R}} \dim_f^V(\omega_V(\hat{B}, t)) \leqslant \sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{M}(t)) \leqslant \frac{-\ln N_\nu}{\ln\left(2(\nu+\lambda)\right) + \gamma_0 \tilde{t}},$$

where $N_\nu = N^W_{\frac{\nu}{\kappa}}(B^V_1(0))$ provided that $(\mathcal{H}_2)$ holds or N_ν comes from (H_2). Here the set $\omega_V(\hat{B}, t)$ with $\hat{B} = \{B(t) \colon t \in \mathbb{R}\}$ is the pullback $\omega-$limit set for the family $\hat{B}$ at time t, i.e.,

$$\omega_V(\hat{B}, t) = \bigcap_{s \leqslant t} \operatorname{cl}_V \bigcup_{r \leqslant s} U(t, r)B(r).$$

The above-mentioned theoretical results are illustrated, among others, by the nonautonomous Chafee-Infante equation already considered in (22), but this time with the Neumann boundary condition

$$\begin{cases} u_t = \triangle_N u + \lambda u - b(t)u^3, \ t > s, \ x \in \Omega, \\ \frac{\partial u}{\partial \vec{n}}(t, x) = 0, \ t > s, \ x \in \partial\Omega, \\ u(s, x) = u_0(x), \ x \in \Omega, \end{cases} \tag{23}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $s \in \mathbb{R}$, $\lambda \geqslant 0$, whereas $\frac{\partial}{\partial \vec{n}}$ denotes the outward unit normal derivative on the boundary $\partial\Omega$.

The key elements for our theory are properties of the nonautonomous term, that is the function $b \colon \mathbb{R} \to (0, \infty)$ of class C^1 such that

(i) $\lim_{t \to -\infty} b(t) = 0$,

(ii) there exists $\beta_1 \in \mathbb{R}$ such that $\frac{b'(t)}{b(t)} \leqslant \beta_1, \ t \in \mathbb{R}$,

(iii) there exist $\gamma_0 > 0$, $K > 0$ and $t_0 \in \mathbb{R}$ such that $b(t) \geqslant K e^{\gamma_0 t}$ for $t \leqslant t_0$.

Note that our assumption is less restrictive than the condition from [C-S2], i.e.,

$$\lim_{t\to-\infty} \frac{e^{\gamma t}}{b(t)} = 0 \text{ for every } \gamma > 0.$$

In particular, in the role of the function b one can take $b(t) = Ke^{\gamma_0 t}$ with some $K, \gamma_0 > 0$ for very negative t and extend b to the right so that (ii) holds.

To justify the existence of the evolution process $\{U(t,s)\colon t \geqslant s\}$ on the space

$$V = X^\alpha \subset \{u \in C^{2\alpha}(\overline{\Omega})\colon \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega\} \text{ with } \frac{1}{2} < \alpha < 1,$$

being the fractional power space corresponding to the operator $-\triangle_N$ considered in $C(\overline{\Omega})$, we prove the a priori estimate in the space $W = C(\overline{\Omega})$. On the other hand, to obtain a pullback absorbing family we apply the method of subsolutions and supersolutions. We verify the assumptions of Theorem 3 and get

Theorem 4 ([3, Theorem 3.3])**.** *The process $\{U(t,s)\colon t \geqslant s\}$ on the space V generated by (23) possesses a pullback exponential attractor $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ in V. In particular, there exists the pullback global attractor $\{\mathcal{A}(t)\colon t \in \mathbb{R}\}$ in V such that for any $\nu \in (0, \frac{1}{2}e^{-\frac{\gamma_0}{2}})$ we have*

$$\mathcal{A}(t) \subset \mathcal{M}(t) = \mathcal{M}^\nu(t) \subset B(t) \subset \widetilde{B}(t),\ t \in \mathbb{R},$$

where

$$\widetilde{B}(t) = \left\{ u \in V\colon \|u\|_W \leqslant \frac{a}{\sqrt{b(t)}} \right\},\ t \in \mathbb{R},$$

with $a > 0$ such that $a^2 \geqslant \lambda + \frac{\beta_1}{2}$, whereas

$$B(t) = \mathrm{cl}_V\, U(t, t-1)\widetilde{B}(t-1),\ t \in \mathbb{R},$$

and

$$\mathrm{diam}_V(B(t)) \leqslant \frac{2a\kappa(t)}{\sqrt{b(t-1)}},\ t \in \mathbb{R} \text{ and } \mathrm{diam}_V(B(t)) \leqslant \frac{2a\kappa(t_0)}{\sqrt{K}} e^{\frac{\gamma_0}{2}} e^{-\frac{\gamma_0}{2}t},\ t \leqslant t_0, \tag{24}$$

where $\kappa\colon \mathbb{R} \to (0,\infty)$ is some nondecreasing function. Moreover, the following estimate

$$\sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{A}(t)) \leqslant \sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{M}^\nu(t)) \leqslant \frac{-\ln N^W_{\frac{\nu}{\kappa(t_0)}}(B_1^V(0))}{\ln(2\nu) + \frac{\gamma_0}{2}}$$

holds.

Observe that if the initial condition u_0 is a positive constant function, then the following function independent of x

$$(U(t,s)u_0)(x) = \frac{e^{\lambda t}}{\sqrt{e^{2\lambda s}u_0^{-2} + 2\int_s^t e^{2\lambda\tau} b(\tau)d\tau}},\ t \geqslant s,\ x \in \overline{\Omega},$$

is a solution of the problem (23). Since $\mathcal{A}(t)$ pullback attracts the singleton $\{u_0\}$ for every $t \in \mathbb{R}$ and $U(t,s)u_0 \to \xi(t)$ in V as $s \to -\infty$, where

$$\xi(t)(x) = \frac{e^{\lambda t}}{\sqrt{2\int_{-\infty}^{t} e^{2\lambda\tau} b(\tau)d\tau}},\ x \in \overline{\Omega},$$

it follows that $\xi(t) \in \mathcal{A}(t)$. The zero solution of the problem (23) also belongs to $\mathcal{A}(t)$, hence

$$\frac{e^{\lambda t}}{\sqrt{2\int_{-\infty}^{t} e^{2\lambda\tau} b(\tau)d\tau}} \leqslant \operatorname{diam}_V(\mathcal{A}(t)) \leqslant \operatorname{diam}_V(\mathcal{M}(t)).$$

If $\lambda > 0$ then it follows from (i) that

$$\operatorname{diam}_V(\mathcal{A}(t)) \to \infty \text{ and } \operatorname{diam}_V(\mathcal{M}(t)) \to \infty \text{ as } t \to -\infty.$$

In the particular case, when $b(t) = Ke^{\gamma_0 t}$, $t \leqslant t_0$, with constants $\gamma_0, K > 0$, we have by (24)

$$\sqrt{\frac{2\lambda+\gamma_0}{2K}}e^{-\frac{\gamma_0}{2}t} \leqslant \operatorname{diam}_V(\mathcal{A}(t)) \leqslant \operatorname{diam}_V(\mathcal{M}(t)) \leqslant \frac{2a\kappa(t_0)}{\sqrt{K}}e^{\frac{\gamma_0}{2}}e^{-\frac{\gamma_0}{2}t},\ t \leqslant t_0,$$

which shows that $\mathcal{A}(t)$ and $\mathcal{M}(t)$ grow exponentially in the past.

In the paper [3] I also consider general reaction-diffusion equations of the form

$$\begin{cases} u_t - \triangle u + f(t,u) = g(t),\ t > s,\ x \in \Omega, \\ u(s,x) = u_0(x),\ x \in \Omega, \quad u(t,x) = 0,\ t > s, x \in \partial\Omega, \end{cases} \tag{25}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$. I assume that $f \in C^1(\mathbb{R}^2, \mathbb{R})$, $g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ and there are constants $p \geqslant 2$, $C_i > 0$, $i = 1, \dots, 5$ such that the growth condition

$$C_1 |u|^p - C_2 \leqslant f(t,u)u \leqslant C_3 |u|^p + C_4,\ u \in \mathbb{R},\ t \in \mathbb{R}, \tag{26}$$

holds and the derivative of the nonlinearity with respect to u is bounded below

$$f_u(t,u) \geqslant -C_5,\ u \in \mathbb{R},\ t \in \mathbb{R}, \quad f(t,0) = 0,\ t \in \mathbb{R}. \tag{27}$$

This problem was investigated in many articles in different aspects. As refers to the existence of the pullback global attractor, it was obtained in the space $H^1_0(\Omega)$ in the paper [L-Z] if f does not depend on time and g has an exponential estimate of the form

$$\|g(t)\|^2_{L^2(\Omega)} \leqslant M_0 e^{\alpha|t|},\ t \in \mathbb{R}, \tag{28}$$

with exponent $0 \leqslant \alpha < \lambda_1$ and $M_0 > 0$, where $\lambda_1 > 0$ is the first eigenvalue of the considered Laplace operator with Dirichlet boundary condition. Later the same result was obtained in the paper [ŁU] (cf. also [SO]) under a more general assumption than (28):

$$\int_{-\infty}^{t} e^{\lambda_1 s} \|g(s)\|^2_{L^2(\Omega)} ds < \infty,\ t \in \mathbb{R}.$$

As refers to the uniform boundedness of the fractal dimension of the pullback global attractor, it was proved in the space $L^2(\Omega)$ in the paper [C-L-V], but only under the additional assumption of f satisfying a global Lipschitz condition with the constant depending on time, i.e., if there exists a positive and nondecreasing function $\xi\colon \mathbb{R} \to (0,\infty)$ such that

$$|f(\tau,u) - f(\tau,v)| \leqslant \xi(t)\,|u-v|\,,\ \tau \leqslant t,\ u,v \in \mathbb{R}, \tag{29}$$

and under the assumption of a power growth of the perturbation g, i.e., if there exist constants $a,b>0$ and $r \geqslant 0$ such that

$$\|g(t)\|_{L^2(\Omega)} \leqslant a\,|t|^r + b,\ t \in \mathbb{R}.$$

In my paper [3] I assumed that f satisfies (26), (27) and (29), whereas g admits even an exponential growth in the assumption (28).

Under these assumptions I verified the requirements of the abstract theory from [3, Corollary 2.6], including the condition (H_2), and proved in [3, Theorem 4.3] that the considered problem (25) generates an evolution process in the space $H^1_0(\Omega)$ (and in $L^2(\Omega)$), which has a pullback exponential attractor $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ in $H^1_0(\Omega)$ with sections being nonempty compact subsets of $H^1_0(\Omega)$ and their diameters are bounded by an exponential function with exponent $\frac{\alpha}{2}\,|t|$. If we choose $\tilde{t} > 0$ and $t_0 \leqslant 0$ and by λ_n denote the sequence of eigenvalues of the Laplace operator considered here, then there exists $n \in \mathbb{N}$ such that the following inequality

$$\lambda := \left(e^{-\lambda_{n+1}\tilde{t}} + \lambda_{n+1}^{-1}\lambda_1^{-1}\xi^2(t_0)e^{2C_5\tilde{t}}\right)^{\frac{1}{2}} < \frac{1}{2}e^{-\frac{\alpha}{2}\tilde{t}} \tag{30}$$

holds. Then we obtain explicitly an estimate of the fractal dimension $\mathcal{M}(t) = \mathcal{M}^\nu(t)$

$$\sup_{t\in\mathbb{R}} \dim_f^{H^1_0(\Omega)}(\mathcal{M}^\nu(t)) \leqslant \frac{-n\ln\left(1 + 2\nu^{-1}e^{\frac{1}{2}\lambda_1^{-1}\xi^2(t_0)\tilde{t}}\right)}{\ln(2(\nu+\lambda)) + \frac{\alpha}{2}\tilde{t}} \tag{31}$$

for any sufficiently small ν. The family $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ pullback attracts at an exponential rate each bounded set in $L^2(\Omega)$ with respect to the Hausdorff semidistance in $H^1_0(\Omega)$. Moreover, the process has the pullback global attractor $\{\mathcal{A}(t)\colon t \in \mathbb{R}\}$ in $H^1_0(\Omega)$ with uniformly bounded fractal dimension by the estimate from (31), which in particular, due to the assumption (28), is a generalization of the results from the paper [C-L-V]. Finally, it is worth mentioning that from the inequality (31) it follows, after further estimating and passing to the limit with ν to 0, that

$$\sup_{t\in\mathbb{R}} \dim_f^{H^1_0(\Omega)}(\mathcal{A}(t)) \leqslant n,$$

where $n \in \mathbb{N}$ satisfies (30).

I presented the results of my studies on pullback exponential attractors, especially the results of my paper [3], during lectures at international scientific conferences in Brazil, Germany and Spain in 2014. I also announced the results during an invited lecture in Centro de Matemática da Universidade do Porto in Portugal in 2013.

Global attractors for impulsive dynamical systems

In 2014 I took a one-month secondment, as part of the program Brazilian-European Partnership in Dynamical Systems (BREUDS), in Instituto de Ciências Matemáticas e de Computação of the University of São Paulo in São Carlos, where together with Alexandre Nolasco de Carvalho and Matheus Bortolan we started to study asymptotics of impulsive dynamical systems. We jointly created a new notion of a global attractor for such systems and together with Everaldo de Mello Bonotto prepared the article [4], which is a part of the scientific achievement.

The theory of impulsive dynamical systems describes phenomena, where the continuous evolution is interrupted by an abrupt change of state. In systems, which we dealt with, these changes are state-dependent and do not occur in explicitly prescribed moments. Possible applications are for instance models of the Lotka-Volterra type with harvest (cull) depending on the state of the population. An inspiration for our studies was the paper [B-D] in which the authors tried to apply the standard definition of a global attractor to problems with impulses ignoring a large class of impulsive dynamical systems.

Mathematical foundations of the mentioned theory come from papers of K. Ciesielski (e.g. [CI1]) and S. Kaul (e.g. [KA]), where the notions of a section and tube, used by us, were formulated. An impulsive dynamical system consists of a continuous semigroup $\{\pi(t)\colon t \geqslant 0\}$ on a metric space (X,d), a nonempty closed subset $M \subset X$, called an impulsive set, such that for every $x \in M$ there exists $\varepsilon_x > 0$ such that

$$F(x,(0,\varepsilon_x)) \cap M = \emptyset \quad \text{and} \quad \bigcup_{t\in(0,\varepsilon_x)} \{\pi(t)x\} \cap M = \emptyset, \tag{32}$$

where

$$F(D,J) = \bigcup_{t\in J} \pi(t)^{-1}(D),\ D \subset X,\ J \subset [0,\infty),$$

and a continuous function $I\colon M \to X$, called an impulsive function. The condition (32) means some kind of transversality of the semigroup in regard to the set M.

We define the function $\phi\colon X \to (0,\infty]$ of the smallest positive time for a point $x \in X$ to reach the set M by

$$\phi(x) = \begin{cases} s & \text{if } \pi(s)x \in M \text{ and } \pi(t)x \notin M \text{ for } 0 < t < s, \\ \infty & \text{if } \bigcup_{t>0}\{\pi(t)x\} \cap M = \emptyset. \end{cases} \tag{33}$$

The impulsive function I is used to construct the impulsive semigroup $\{\tilde\pi(t)\colon t \geqslant 0\}$ on X which, apart from satisfying the conditions

$$\tilde\pi(t+s) = \tilde\pi(t)\tilde\pi(s),\ t,s \geqslant 0,\ \tilde\pi(0) = Id,$$

is usually discontinuous. An impulsive trajectory $t \mapsto \tilde\pi(t)x$ of a point $x \in X$ coincides with the trajectory $t \mapsto \pi(t)x$ until it reaches the set M for the first time. Then a jump occurs in accordance with the function $I\colon M \to X$ and from the new point the evolution follows the semigroup $\{\pi(t)\colon t \geqslant 0\}$ again until it reaches the set M anew, and so on (cf. the details in [4]). In the paper we assume that the semigroup $\{\tilde\pi(t)\colon t \geqslant 0\}$ is well-defined, which is true for example if $\phi(z) \geqslant \xi > 0$ for every $z \in I(M)$.

Even a simple example [4, Example 1.5] shows that the requirements regarding the attractor from the paper [B-D] are too restrictive. We introduce a new definition of the global attractor for an impulsive dynamical system (X,π,M,I).

Definition 5 ([4, Definition 1.6])**.** A subset $\mathcal{A} \subset X$ is called the global attractor for an impulsive dynamical system (X, π, M, I) if it satisfies the following conditions:

(i) $\mathcal{A}$ is precompact (i.e., its closure in X is compact) and $\mathcal{A} = \mathrm{cl}_X \mathcal{A} \setminus M$,

(ii) $\mathcal{A}$ is $\tilde{\pi}-$invariant, i.e., $\tilde{\pi}(t)\mathcal{A} = \mathcal{A}$, $t \geqslant 0$,

(iii) $\mathcal{A}$ $\tilde{\pi}-$attracts every bounded set B in X, i.e.,

$$\mathrm{dist}_X(\tilde{\pi}(t)B, \mathcal{A}) = \sup_{x\in B} \inf_{y\in\mathcal{A}} d(\tilde{\pi}(t)x, y) \to 0 \text{ as } t \to \infty.$$

It is easy to see that such an object is uniquely determined. Moreover, by this definition the system from [4, Example 1.5] possesses the global attractor with unusual properties in comparison with continuous semigroups: the attractor is not connected, consists of two isolated invariant sets – a periodic orbit and a stationary point, without a connecting orbit in the attractor, whereas orbits reach the periodic orbit in a finite time. This makes such objects worth studying more thoroughly.

The aim of the paper [4] was to create counterparts of known classical theorems for global attractors from Definition 5 for impulsive dynamical systems.

We showed in [4, Proposition 4.3] that the attractor consists of points through which passes a bounded complete $\tilde{\pi}-$orbit, which corresponds to [C-D, Corollary 1.1.1 (iii)].

Similarly in [4, Proposition 4.4] we proved that the global attractor is the union of impulsive $\omega-$limit sets (cf. [4, Definition 3.1 and Lemma 3.2]) of bounded subsets of X with points of M removed. This is a counterpart of the result [C-D, Corollary 1.1.1 (i)].

In [4, Proposition 4.5] we noticed that the global attractor for an impulsive dynamical system is minimal among all subsets $K \subset X$ satisfying $K = \mathrm{cl}_X K \setminus M$ which $\tilde{\pi}-$attract all bounded subsets of X, which resembles the observation [C-D, Observation 1.1.3].

The theorem [4, Theorem 4.7] is the main result of the paper in which we prove the existence of the global attractor for strongly dissipative impulsive dynamical systems. It constitutes a counterpart of the theorem [RO, Theorem 10.5].

Definition 6 ([4, Definition 4.6])**.** An impulsive dynamical system (X, π, M, I) is called strongly dissipative if there exists a nonempty precompact set K in X such that $K \cap M = \emptyset$, which $\tilde{\pi}$-absorbs all bounded sets in X, i.e., for any bounded set B in X there exists $t_B \geqslant 0$ such that $\tilde{\pi}(t)B \subset K$ for all $t \geqslant t_B$.

Theorem 7 ([4, Theorem 4.7])**.** *Let (X, π, M, I) be a strongly dissipative impulsive dynamical system with the $\tilde{\pi}-$absorbing precompact set K, satisfying $I(M) \cap M = \emptyset$ and such that every point of M satisfies the condition (SSTC) (see the details in [4, Definition 2.3]) and $\phi(z) \geqslant \xi > 0$ holds for all $z \in I(M)$. Then (X, π, M, I) possesses the global attractor $\mathcal{A}$ and we have*

$$\mathcal{A} = \tilde{\omega}(K) \setminus M. \tag{34}$$

The proof of Theorem 7 comes down to the verification of the properties from Definition 5 for the set given in (34). It is not, however, an easy task and requires development of the theory of impulsive $\omega-$limit sets, which was done in the third, fundamental section of the paper [4]. The first step consisted in proving the positive invariance of sets of the form $\tilde{\omega}(B) \setminus M$, which was done in [4, Proposition 3.7] under the assumption that the points of M satisfy the so called strong tube condition (STC) coming from papers of K. Ciesielski and S. Kaul (cf. [4, Definition 2.3]). This condition guarantees among others the upper semicontinuity of the function ϕ in X and its continuity in $X \setminus M$ (cf. [CI2, Theorem 3.8]). It is more difficult to show the negative invariance of sets of the form $\tilde{\omega}(B) \setminus M$. To this end, I suggested introducing the special strong tube condition (SSTC) from [4, Definition 2.3], which helped to prove the negative invariance ([4, Proposition 3.12]) as well as $\tilde{\pi}-$attraction by sets of the form $\tilde{\omega}(B) \setminus M$ ([4, Proposition 3.14]).

The last fourth section of the paper [4] was written together with Everaldo de Mello Bonotto and contains an illustration of the presented theory for systems of ordinary differential equations on the plane, autonomous systems in $\mathbb{R}^N$ and for nonlinear reaction-diffusion equations with Dirichlet boundary condition. The problem for planar systems, although very simple, shows that our abstract assumptions can be verified in concrete cases. For systems of ordinary differential equations in $\mathbb{R}^N$ as well as initial boundary value problems for reaction-diffusion equations of the form

$$\begin{cases} u_t - \Delta u = f(u), \ t > 0, \\ u|_{\partial\Omega} = 0, \ t > 0, \quad u(0) = u_0 \in L^2(\Omega), \end{cases}$$

with an impulsive function, we formulate sufficient conditions for the generated impulsive dynamical systems to be strongly dissipative and in accordance with Theorem 7 to possess the global attractors in the sense of Definition 5.

I presented the results of the paper [4] described above during my lectures at the VII Symposium on Nonlinear Analysis at the Nicolaus Copernicus University in Toruń in 2015 and during the 6th IST-IME Meeting in Lisbon in 2016. I also presented them during my invited lecture at the Faculty of Mathematics, Informatics and Mechanics at the University of Warsaw in 2015.

The paper [4] lays foundations for the study of structural stability for impulsive dynamical systems. In the paper [12] we show that, under certain collective tube conditions, the global attractors $\mathcal{A}_\eta$, $\eta \in [0,1]$, for the family of impulsive dynamical systems $(X, \pi_\eta, M_\eta, I_\eta)$, $\eta \in [0,1]$, are upper semicontinuous at $\eta = 0$, i.e.,

$$\lim_{\eta \to 0} \operatorname{dist}_X(\mathcal{A}_\eta, \mathcal{A}_0) = 0. \tag{35}$$

Aspects of structural stability

The paper [5], which is a part of the scientific achievement, was written by me together with Carlos Rocha during my postdoctoral fellowship in the Instituto Superior Técnico in Lisbon in years 2005-2007. It concerns the transversality of stable and unstable manifolds of hyperbolic periodic orbits for scalar reaction-diffusion equations with periodic boundary condition.

The notion of transversality of invariant manifolds of critical elements plays an important role in theorems on generic properties, i.e., the ones which hold on a residual set, that is, a countable intersection of open dense sets. As an example serves here the Kupka-Smale theorem (cf. [P-M, Chapter 3], [SZ, Twierdzenie 3.7.1]) on the residuality of the set of the so called Kupka-Smale vector fields in the space of C^r, $r \geqslant 1$, vector fields on a compact differential manifold. Let us recall that a vector field X on a compact differential manifold has the Kupka-Smale property if all critical points and periodic orbits for X are hyperbolic and their invariant manifolds intersect transversally. The transversality of invariant manifolds of critical elements is also the foundation of theorems on structural stability. We say that a vector field X of class C^r, $r \geqslant 1$, on a compact differential manifold M is structurally stable if there exists a neighborhood U of the vector field X such that each vector field $Y \in U$ is topologically equivalent to X, that is, there exists a homeomorphism of the manifold M, which takes orbits of X into orbits of Y preserving their orientation. We say that (cf. [P-M, Chapter 4]) a vector field X of class C^r, $r \geqslant 1$, on M is the Morse-Smale vector field if X has finitely many critical elements all of them being hyperbolic, their invariant manifolds intersect transversally and the set of nonwandering points is the union of all critical elements. J. Palis and S. Smale proved in [PA], [P-S] that Morse-Smale vector fields form an open set in the space of C^r, $r \geqslant 1$, vector fields on M and are structurally stable.

For semigroups the theorem on the A-structural stability of the Morse-Smale semigroups was proved by W.M. Oliva (cf. [OL], [H-M-O, Chapter 6]). It is assumed that $\mathcal{F}$ is a topological space of parameters for semigroups $\{T_f(t)\colon t \geqslant 0\}$ of class C^k on a Banach space X, which possess global attractors $\mathcal{A}_f$ in X, the mapping $\mathcal{F} \ni f \mapsto \mathcal{A}_f \in X$ is upper semicontinuous and $T_f(t)|_{\mathcal{A}_f}$ and $DT_f(t)|_{\mathcal{A}_b(DT_f)}$ are one-to-one for $t \geqslant 0$. Among $f \in \mathcal{F}$ we distinguish $f \in MS$ for which the semigroup $\{T_f(t)\colon t \geqslant 0\}$ on X is the Morse-Smale semigroup, i.e., it has finitely many critical elements (stationary points and periodic orbits) all of them hyperbolic, their union forms the set of nonwandering points and the unstable manifolds of critical elements have finite dimension and intersect transversally with local stable manifolds. Then the set MS is open in $\mathcal{F}$ and each $f \in MS$ is A-structurally stable, i.e., there exists a neighborhood U of the parameter $f \in \mathcal{F}$ such that for every $g \in U$ we have a homeomorphism $h = h(g)\colon \mathcal{A}_f \to \mathcal{A}_g$ which takes the orbits in $\mathcal{A}_f$ to orbits in $\mathcal{A}_g$ preserving their orientation.

One of the first examples of infinite-dimensional Morse-Smale semigroups generated by partial differential equations come from the papers of D. Henry [HE2] and S. Angenent [AN1] and concern scalar semilinear parabolic equations with boundary conditions of Dirichlet or Neumann type for which the only critical elements are stationary solutions. In particular, this also refers to the Chafee-Infante equation. The problem of the automatic transversality of intersections of invariant manifolds for stationary points was also considered in the paper [C-C-H] for the nonautonomous (periodic in time) scalar semilinear parabolic equation with nonlinearity depending only on time, the space variable and the solution with the homogeneous Dirichlet boundary condition as well as for problems in symmetric domains in $\mathbb{R}^N$ in the paper [PO]. In the paper [F-O] the automatic transversality of invariant manifolds for stationary points and periodic orbits was proved in the case of special classes of systems of ordinary differential equations.

In our paper [5] we consider the following scalar reaction-diffusion equation with periodic boundary conditions and the nonlinearity $f\colon S^1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ depends on $x \in S^1$, the solution u and its spatial derivative, i.e.,

$$\begin{cases} u_t = u_{xx} + f(x, u, u_x),\ x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, \\ u(0,x) = u_0(x),\ x \in S^1. \end{cases} \tag{36}$$

We assume that f is C^2 and has subquadratic growth with respect to u_x, i.e.,

there exists $0 \leqslant \gamma < 2$ and a continuous function $k\colon [0,\infty) \to [0,\infty)$ such that
$$|f(x,y,z)| \leqslant k(r)(1 + |z|^\gamma),\ (x,y,z) \in S^1 \times [-r,r] \times \mathbb{R} \text{ for every } r > 0,$$

and it satisfies the dissipativity condition

$$yf(x,y,0) < 0,\ (x,y) \in S^1 \times \mathbb{R},\ |y| \geqslant K \text{ for some } K > 0.$$

If we fix $\alpha \in (\frac{3}{4}, 1)$ then the fractional power space $X^\alpha = H^{2\alpha}(S^1)$ corresponding to the operator $Au = -u_{xx} + u$ in $L^2(S^1)$ is embedded into the space of C^1 functions on a circle. Under the assumptions on f each X^α solution of the problem (36) exists globally in time and we can construct the semigroup $\{T(t)\colon t \geqslant 0\}$ of global X^α solutions. This semigroup is compact and point dissipative, so it possesses the global attractor $\mathcal{A}$ in X^α, which is the union of all bounded complete orbits, i.e., defined on $\mathbb{R}$. Moreover, in [5, Section 3] we show that every operator $T(t)$ is injective, the linearization along each solution defines a linear evolution process on X^α formed by compact bounded operators and each such operator is injective and has a dense range.

Having the global attractor we would like to know its structure. In the papers [A-F] and [F-R-W] it was proved that if the nonlinearity f does not depend explicitly on x, then the attractor consists exclusively of stationary points and the so called rotating waves, i.e., orbits of periodic solutions of the form

$$u(t,x) = v(x - ct),\ t \in \mathbb{R},\ x \in S^1 \text{ with some } c \neq 0,$$

and heteroclinic connections between these critical elements, all of them assumed to be hyperbolic. However, from the paper [S-F] it follows that in the case when the nonlinearity f depends explicitly on x, in the attractor may also appear homoclinic orbits. The first aim of our paper [5] was to prove that such a homoclinic connection cannot occur for a hyperbolic periodic orbit. The other aim was to prove that if we have two distinct hyperbolic periodic orbits, then their stable and unstable manifolds intersect transversally, i.e., the tangent spaces to these manifolds at each point of intersection complement to the whole space X^α.

The main tool used in the proofs is the theory of the so called zero number of a C^1 function developed by H. Matano [MA] (see also [B-F], [AN2]) and connected with classical results of C. Sturm [ST]. If we denote by $z(\varphi)$ the (even) number of strict sign changes of a C^1 function $\varphi\colon S^1 \to \mathbb{R}$, then (see [5, Lemmas 3.1, 3.2] and [M-N, Lemmas 3.2, 3.4]) for two different X^α solutions of the problem (36) on an interval J their difference $v(t) = u_1(t) - u_2(t)$, $t \in J$, satisfies the conditions:

(i) $z(v(t))$ is finite for every $t \in J$,

(ii) $z(v(t))$ is a nonincreasing function of t on J,

(iii) $z(v(t))$ strictly decreases at $t = t_0$ if and only if there exists $x_0 \in S^1$ such that

$$v(t_0)(x_0) = 0,\ \partial_x v(t_0)(x_0) = 0.$$

The other important element of our considerations are spectral properties of the so called period map (cf. [HE1, Definition 7.2.1], [5, Section 4]) T_ω for the linearization of our equation along a periodic solution p with period $\omega > 0$ which determines the periodic orbit Π. The operator T_ω corresponds to the monodromy matrix for linear ordinary differential equations with periodic coefficients. Similarly to the classical theory, its eigenvalues are called characteristic multipliers. If we put them in a sequence $\{\lambda_j\}_{j\geqslant 0}$ counting algebraic multiplicities and ordering them by the inequality $|\lambda_{j+1}| \leqslant |\lambda_j|$, then by the paper [A-F] we know that $|\lambda_{2j+1}| < |\lambda_{2j}|$ for all $j \geqslant 0$. In other words, denoting by $E_j(\Pi)$ the real generalized eigenspace corresponding to $\{\lambda_{2j-1}, \lambda_{2j}\}$ for $j \geqslant 1$ and by $E_0(\Pi)$ the real eigenspace corresponding to the isolated eigenvalue λ_0, we know that $\dim E_0(\Pi) = 1$ and $\dim E_j(\Pi) = 2$, $j \geqslant 1$. Moreover, it follows from [A-F, Theorem 2.2] that any nonzero function $\phi \in E_j(\Pi)$, $j \geqslant 0$, has only simple zeros and $z(\phi) = 2j$.

We distinguish the elements of the spectrum lying inside the unit circle, on the unit circle and outside the unit circle. The number of the latter ones counted with multiplicities we denote by $i(\Pi)$ and call the Morse index of the periodic orbit Π.

We study the situation where the periodic orbit Π is hyperbolic, that is, the corresponding period map T_ω has 1 as a characteristic multiplier which is a simple eigenvalue and unique on the unit circle. Then the corresponding eigenfunction is $p_t(0)$.

First we consider the local stable manifold $W^s_{loc}(\Pi)$ for a hyperbolic periodic orbit Π with period $\omega > 0$. The main result of this part is [5, Theorem 5.2].

Theorem 8. *For an initial condition $u_0 \in W^s_{loc}(\Pi) \setminus \Pi$ there exist a phase $a \in \Pi$ and $\kappa > 0$ such that*

$$\lim_{t\to\infty} e^{\kappa t} \left\| T(t)u_0 - T(t)a \right\|_{X^\alpha} = 0$$

and, for $2N = z(p_t(0; a))$ with $p(t; a) = T(t)a$, we have

$$z(u_0 - a) \geqslant \begin{cases} i(\Pi) + 1 = 2N & \text{if } i(\Pi) = 2N - 1, \\ i(\Pi) + 2 = 2N + 2 & \text{if } i(\Pi) = 2N. \end{cases} \tag{37}$$

Next, we study the unstable manifold $W^u(\Pi)$ and similarly as before we obtain the following result in [5, Theorem 6.2].

Theorem 9. *For any $u_0 \in W^u(\Pi) \setminus \Pi$ there exist a phase $a \in \Pi$ and $\kappa' > 0$ such that*

$$\lim_{t\to\infty} e^{\kappa' t} \left\| T(t)^{-1}u_0 - T(t)^{-1}a \right\|_{X^\alpha} = 0$$

and, for $2N = z(p_t(0; a))$ with $p(t; a) = T(t)a$, we have

$$z(u_0 - a) \leqslant \begin{cases} i(\Pi) - 1 = 2N - 2 & \text{if } i(\Pi) = 2N - 1, \\ i(\Pi) = 2N & \text{if } i(\Pi) = 2N. \end{cases} \tag{38}$$

Both these results lead to the exclusion of existence of a homoclinic orbit for a hyperbolic periodic orbit. Indeed, consider two (not necessarily distinct) hyperbolic periodic orbits Π^- and Π^+ with periods $\omega^- > 0$ and $\omega^+ > 0$, respectively. We assume that there exists a point

$$u_0 \in (W^u(\Pi^-) \cap W^s_{loc}(\Pi^+)) \setminus (\Pi^- \cup \Pi^+).$$

In particular, if $\Pi^- = \Pi^+$ then u_0 lies on a homoclinic orbit for the periodic orbit. It follows from the above-mentioned theorems that there exist corresponding phases $a^\pm \in \Pi^\pm$ and corresponding periodic solutions $p^\pm$ such that

$$\left\| u(t; u_0) - p^+(t; a^+) \right\|_{X^\alpha} \to 0 \text{ and } \left\| u(-t; u_0) - p^-(-t; a^-) \right\|_{X^\alpha} \to 0 \text{ as } t \to \infty.$$

Furthermore, we have

$$z(u_0 - a^-) \geqslant z(u_0 - a^+),$$

which together with the estimates (37) and (38) for the orbits $\Pi^\pm$ implies one of the two main results of the paper [5].

Theorem 10 ([5, Theorem 7.3]). *Denoting* $2N^{\pm} = z(p_t^{\pm}(0; a^{\pm}))$, *the following inequalities*

$$N^- \geqslant N^+ \quad \textit{and} \quad i(\Pi^-) \geqslant i(\Pi^+) + 1$$

hold, which excludes the existence of a homoclinic connection for a hyperbolic periodic orbit for (36). *Moreover, if* $i(\Pi^+) = 2N^+$ *then we even have*

$$N^- \geqslant N^+ + 1.$$

The other main result of the paper [5] is the automatic transversality of intersection of the unstable manifold and stable manifold for two distinct hyperbolic periodic orbits for the problem (36) obtained in [5, Theorem 8.2].

Theorem 11. *The stable and unstable manifolds for two hyperbolic periodic orbits* $\Pi^{\pm}$ *for the problem* (36) *have a transversal intersection*

$$W^u(\Pi^-) \pitchfork W^s_{loc}(\Pi^+),$$

i.e., if $u_0 \in T(\tau)W^u_{loc}(\Pi^-) \cap W^s_{loc}(\Pi^+)$ *for some* $\tau \geqslant 0$, *then*

$$T_{u_0}T(\tau)W^u_{loc}(\Pi^-) + T_{u_0}W^s_{loc}(\Pi^+) = X^{\alpha}.$$

The proof is based on ideas coming from the paper [C-C-H]. We denote the respective phases by $a^{\pm}$ as above and the periodic solutions by $p^{\pm}$. Let $2N^{\pm}$ be the zero numbers of eigenfunctions corresponding to the characteristic multipliers 1. In the proof we use the following observation: for nonzero functions from the tangent subspace at u_0 to the local stable manifold for Π^+ the zero number is greater or equal to $2N^+$. Moreover, there exists a subspace W^+ of the tangent space at u_0 to the local stable manifold of the orbit Π^+ with codimension $2N^+ + 1$ and the zero number greater or equal to $2N^+ + 2$. Next, we consider two cases according to the Morse index for the orbit Π^+. Let $i(\Pi^+) = 2N^+ - 1$. Then the codimension of the local stable manifold is $2N^+ - 1$. Since by Theorem 10 we have $N^- \geqslant N^+$, we know that the direct sum $E_0 \oplus \ldots \oplus E_{N^+-1}$ of the subspaces corresponding to the point a^- is a subspace of the tangent space to the unstable manifold for Π^- at a^-. Moreover, the zero number for nonzero functions from this space does not exceed $2N^+ - 2$. Since we are on a C^1 submanifold of X^{α} and the sequence $u(-m\omega^-)$ converges to a^- as $m \to \infty$, there exists for large m a subspace of the tangent space at $u(-m\omega^-)$ to the unstable manifold with the same dimension and the same zero number estimate. Furthermore, we move to the point u_0 using the evolution operators for the linearization along the connecting solution u. Since these operators are one-to-one and do not increase the zero number, we obtain a subspace of the tangent space to the unstable manifold at u_0 with dimension equal to the codimension of the local stable manifold. Additionally, this space has a trivial intersection with the tangent space to the stable manifold at u_0 due to the zero number estimates. This shows that in this case the tangent spaces complement to the whole space X^{α}.

The analysis of the case $i(\Pi^+) = 2N^+$ is similar, but this time we use the inequality $N^- \geqslant N^+ + 1$ from Theorem 10 and from the tangent space to the local stable manifold at u_0 we choose the subspace W^+ with codimension $2N^+ + 1$. This again implies that the tangent spaces complement to the whole space X^{α} also in this case.

I presented the above-mentioned results of the paper [5] among others at the conference ICMC Summer Meeting on Differential Equations 2008 Chapter at the Universidade de

São Paulo in São Carlos in Brazil. During this conference R. Joly delivered a lecture on the results of his joint research with G. Raugel, published later in the paper [J-R1], in which he presented the proof of genericity of hyperbolic stationary points and periodic orbits for scalar parabolic equations of the form (36) with f from the space $\mathcal{F} = C^2(S^1 \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ endowed with the Whitney topology (see [J-R1, Theorem 1.2]). In the joint discussion of four authors we noticed that the argument which I described above can be immediately applied to the case of the intersection of stable and unstable manifolds if one of the critical elements is a hyperbolic stationary point and the other one is a hyperbolic periodic orbit. Thus also in this case the automatic transversality of invariant manifolds holds, which was later published in [J-R2, Theorem 4.2]. In [J-R2] the authors also proved for (36) the automatic transversality of invariant manifolds of hyperbolic stationary points with distinct Morse indices and the fact that the absence of connections between stationary points with the same Morse index is a generic property. Consequently, this leads to the genericity of the Morse-Smale property for scalar parabolic equations on a circle of the form (36) proved in [J-R2, Theorem 1.6].

5. Other scientific research achievements

List of other papers and monographs:

[6] Radosław Czaja, *Differential Equations with Sectorial Operator*, Wydawnictwo Uniwersytetu Śląskiego, Katowice, 2002, ISBN 83-226-1164-1.

[7] Radosław Czaja, Dynamically equivalent perturbations of linear equations, *Demonstratio Mathematica* 37 (2004), 327–348.

[8] Radosław Czaja, Asymptotics of parabolic equations with possible blow-up, *Colloquium Mathematicum* 99 (2004), 61–73.

[9] Jan W. Cholewa, Radosław Czaja, Gianluca Mola, Remarks on the fractal dimension of bi-space global and exponential attractors, *Bollettino dell'Unione Matematica Italiana* (9) 1 (2008), 121–145.

[10] Radosław Czaja, Messoud Efendiev, A note on attractors with finite fractal dimension, *Bulletin of the London Mathematical Society* 40 (2008), 651–658.

[11] Radosław Czaja, Bi-space pullback attractors for closed processes, *São Paulo Journal of Mathematical Sciences* 6, 2 (2012), 227–246.

[12] Everaldo de Mello Bonotto, Matheus Cheque Bortolan, Rodolfo Collegari, Radosław Czaja, Semicontinuity of attractors for impulsive dynamical systems, *Journal of Differential Equations* 261 (2016), 4338–4367.

[13] Radosław Czaja, Pedro Marín-Rubio, Pullback exponential attractors for parabolic equations with dynamical boundary conditions, *Taiwanese Journal of Mathematics*, article in press, available online since December 20, 2016 at
`http://journal.tms.org.tw/~journal/tjm/201612/m455-7862-preview.pdf`

The monograph [6] is an extension of my master thesis which won a commendation in the Józef Marcinkiewicz competition organized by the Polish Mathematical Society. It is devoted to the theory of strongly continuous linear semigroups with special emphasis on analytic semigroups and their role in the modern approach to differential equations. It contains an original lecture concerning generation of analytic semigroups by unbounded linear operators in Banach spaces. Especially worthy of note is the precision of the theorem ([6, Theorem 2.2.7]) on the necessary and sufficient condition for a linear operator to generate analytic semigroup. In the subsequent part of the publication I present the theory of powers of positive sectorial operators (cf. [HE1], [AM], [LU]) laying the good foundations to study the existence and uniqueness of solutions of abstract differential equations. Following the monographs [PZ], [HE1] and [C-D], I present in [6, Chapter 4] the theory of the existence and uniqueness of solutions of nonhomogeneous linear equations in Banach spaces and semilinear equations with a sectorial operator in the main part with the right-hand side which is nonlinear with respect to the solution we are looking for. The latter type of equations is particularly important from the point of applications to parabolic semilinear systems of partial differential equations arising from the mathematical physics as for example the renowned Navier-Stokes system. This is the type of equations to which I later devoted most of my scientific career. I am glad that this monograph, though with scarce circulation, helped master and Ph.D. students in their further individual research in Poland as well as abroad (see e.g. [SI], [A-O]).

In the paper [7] I studied nonlinear autonomous perturbations of abstract parabolic equations with a sectorial operator A of the form

$$u_t + Au = F_\lambda(u),\ t > 0, \tag{39}$$

for which there exist global attractors $\mathcal{A}_\lambda$ independent of the parameter λ. This problem refers to the notion of synchronized semigroups $\{T_\lambda(t)\colon t \geqslant 0\}$ for (39) considered in the paper [HA2]. The key condition of my main result [7, Theorem 2.6], ensuring the existence of the common attractor for the family of autonomous equations (39), was to indicate continuous functions distinguishing stationary solutions and constant along trajectories. I verified the assumptions of the mentioned theorem in the case of a scalar second order equation with Neumann boundary condition ([7, Example 3.1]) and in the case of the Cahn-Hilliard system modeling phase separation of a multi-component alloy:

$$\begin{cases} u_t(t,x) = -\triangle\left[\Gamma\triangle u(t,x) - \nabla_u\lambda(u(t,x))\right],\ t > 0,\ x \in \Omega, \\ \nabla u(t,x)\vec{n}(x) = \nabla(\Delta u(t,x))\vec{n}(x) = 0,\ t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x),\ x \in \Omega, \end{cases}$$

where $u\colon [0,\infty) \times \overline{\Omega} \to \mathbb{R}^m$, $u^T = (u_1, \dots, u_m)$, $\Gamma \in \mathbb{R}^{m\times m}$ is a positively defined symmetric matrix and Ω is a bounded domain in $\mathbb{R}^N$ with $N \leqslant 3$ and the boundary $\partial\Omega$ of class $C^{4+\varepsilon}$. In the considered problem the role of parameters λ is played by functions of class $C^{3+Lip}(\mathbb{R}^m)$, which are bounded below and convex. Then the attractors $\mathcal{A}_\lambda$ coincide with the attractor $\mathcal{A}_0$ for the linear problem and consist of all functions constant almost everywhere in Ω with absolute value not exceeding some constant conditioned by the considered phase space (cf. [7, Example 3.3]).

The paper [8] concerns the description of the asymptotic behavior of solutions admitting blow-up of some of them for the problem

$$\begin{cases} u_t + Au = F(u),\ t > 0, \\ u(0) = u_0, \end{cases} \tag{40}$$

where A is a sectorial operator with a compact resolvent and the right-hand side $F\colon X^\alpha \to X$ fulfills the Lipschitz condition on bounded subsets of the fractional power space X^α corresponding to the operator A. I admit the situation where for some initial data u_0 the norm of the solution in the space X^α becomes unbounded in finite of infinite time. I introduce the natural phase space

$$V = \{u_0 \in X^\alpha\colon \sup_{t\in[0,\infty)} \|u(t;u_0)\|_{X^\alpha} < \infty\},$$

I assume that $V \neq \emptyset$ and define the semigroup $\{T(t)\colon t \geqslant 0\}$ of global X^α solutions of (40) on the space V. Despite the compactness of the resolvent of the operator A, the compactness of this semigroup is not guaranteed, since we do not know a priori whether V is a closed subset of X^α. Nevertheless, my assumptions allow to prove that the $\omega-$limit sets of sets with bounded positive orbits are nonempty, compact and invariant.

I consider the set $\mathcal{S}$ of all points from V through which passes a bounded complete orbit for the semigroup $\{T(t)\colon t \geqslant 0\}$. In [8, Theorem 2.5] I prove that $\mathcal{S}$ is nonempty, invariant and attracting every subset of V with bounded positive orbit. Moreover, if $\mathcal{S}$ is bounded then $\mathcal{S}$ is a compact set and a maximal bounded and invariant set. If we assume additionally that positive orbits of bounded sets in V are bounded or V is a closed subset of X^α, then $\mathcal{S}$ is the global attractor for the semigroup $\{T(t)\colon t \geqslant 0\}$ in V. Hence I conclude ([8, Corollary 2.6]) that if all bounded complete orbits of points are uniformly bounded in X^α, then every X^α solution of the problem (40) either blows up in finite or infinite time or stays bounded in X^α and is attracted by a maximal compact and invariant set. These results are illustrated by two examples. The first of them (cf. [8, Example 3.1]) is the Frank-Kamenetskii equation with Dirichlet boundary condition being a model in the combustion theory

$$\begin{cases} u_t = \Delta u + \lambda e^u,\ t > 0,\ x \in B_1(0) \subset \mathbb{R}^N, \\ u(t,x) = 0,\ t > 0,\ x \in \partial B_1(0), \quad u(0,x) = u_0(x),\ x \in B_1(0). \end{cases} \tag{41}$$

It follows from the paper [FU] that for small $\lambda > 0$ and some initial conditions the solution of (41) blows up in finite time. Considering the one-dimensional case I justified in my paper that the solutions which stay bounded are attracted by the maximal compact and invariant set consisting of two stationary solutions and a heteroclinic orbit connecting both the solutions. The other example concerns the set $\mathcal{S}$ for the $N-$dimensional Navier-Stokes system for a viscous incompressible fluid under a small external force (cf. [8, Example 3.2]).

In the paper [9] together with Jan W. Cholewa we studied applications of the smoothing property to estimating fractal dimension of the bi-space global attractor, which is a generalization of the classical notion of the global attractor considered e.g. in the monograph [B-V], and to the introduction of the notion of a bi-space exponential attractor. In [9, Lemma 2.1] we showed some generalization of the known lemma [M-P, Lemma 1.3] to estimate fractal dimension of negatively invariant sets, which, together with the result on the existence of the bi-space global attractor ([9, Corollary 2.3]), leads to theorems [9, Theorem 2.5, Corollary 2.6] on the existence of the bi-space global attractor with finite fractal dimension. Referring to the construction from the paper [E-M-Z], we proved in [9, Proposition 2.7, Corollary 2.8] the existence of a bi-space exponential attractor for dissipative semigroups which can be decomposed for large times into a contracting part in a weak space and into the smoothing part on the absorbing set. Moreover, in [9, Corollary

2.9] the existence of an exponential attractor was proved for the semigroup satisfying the smoothing property.

The examples contained in [9] are the essential element of the paper. In the first of them we show the existence of an $(X^\alpha - X^\beta)$ exponential attractor with $\beta \in (\alpha, 1)$ for the abstract semilinear problem of the form (40) with a sectorial operator A with a compact resolvent, which contains the $(X^\alpha - X^\beta)$ global attractor of finite fractal dimension (cf. [9, Theorem 3.2]). Further examples concern specific problems for partial differential equations. We apply the above-mentioned theorem to reaction-diffusion equations with subquadratic growth for the gradient and to the wave equation with the damping operator $(-\Delta_D)^{\frac{1}{2}}$. Moreover, applying [9, Corollary 3.3] which is a corollary of [9, Lemma 2.1], we derive estimates of the fractal dimension of the global attractor for the Cahn-Hilliard equation ([9, Corollary 3.9]) and the global attractor for higher order parabolic equations with elliptic operators of order $2m$ in the main part ([9, Corollary 3.10]). An interesting application of [9, Lemma 2.1] is its use in [9, Theorem 3.8] to estimating the fractal dimension of the global attractor in the space $H_0^1(\Omega) \times L^2(\Omega)$ for the wave equation with the damping operator $(-\Delta_D)$

$$\begin{cases} u_{tt} - \Delta_D u_t - \Delta_D u = f(u), \ t > 0, \ x \in \Omega \subset \mathbb{R}^3 \text{ bounded }, \\ u(0,x) = u_0(x), \ u_t(0,x) = v_0(x), \ x \in \Omega, \\ u(t,x) = 0, \ t \geqslant 0, \ x \in \partial\Omega, \end{cases} \tag{42}$$

with the dissipative nonlinearity $f \in C^2(\mathbb{R}, \mathbb{R})$ satisfying the critical growth condition

$$\exists_{c>0} \ |f''(s)| \leqslant c(1 + |s|^3), \ s \in \mathbb{R}.$$

The difficulty in this case lies in the noncompactness of the resolvent of the sectorial operator for the abstract Cauchy problem generated by (42). The last example from the paper [9] was written by Gianluca Mola and concerns the existence of an exponential attractor, containing the finite-dimensional global attractor, for a nonparabolic problem with memory.

During my postdoctoral fellowship in the Instituto Superior Técnico in Lisbon I met Messoud Efendiev from the Helmholtz Center Munich in Germany and started the scientific collaboration with him. Its first effect was the paper [10] in which we created a tool ([10, Theorem 2.1]) to estimate the fractal dimension of an invariant set, e.g. the global attractor. It is based on the smoothing property again, but instead of compactly embedding the smoother space into the base space, we assume here that it is compactly embedded into some auxiliary space Z. This result is a generalization of the results regarding estimates of the dimension of the attractor from the papers [M-P, Lemma 1.3] and [E-M-Z, Proposition 1]. Moreover, we show in [10, Theorem 3.2] that identical estimates on a positively invariant set allow to prove the existence of an exponential attractor for a mapping as well as for a semigroup, after adding supplementary appropriate condition on its continuity (see [10, Corollary 3.5]). The scientific contact with Messoud Efendiev became fruitful in the form of other two joint papers [1, 2] on exponential attractors for nonautonomous problems, which have been described in the part regarding the scientific achievement.

The existence of the global attractor for a semigroup is shown using some kind of its continuity. It turns out, however, that it suffices to assume closedness for this purpose (cf. [P-Z]) or even only asymptotic closedness of the operators forming the semigroup (cf. [C-R]). These approaches were an inspiration to write my paper [11] on the pullback global attractors for closed evolution processes. This paper is a study of conditions which lead to the existence of the bi-space $(V-W)$ pullback global $\mathcal{D}-$attractor, which attracts all elements of some universe $\mathcal{D}$, i.e., distinguished families of sets $\hat{D}=\{D(t)\colon t\in\mathbb{R}\}$ (cf. [11, Definition 2.2]). In the classical framework the universe $\mathcal{D}$ is made of families given by all bounded sets in the phase space V.

First I investigate equivalent statements for the asymptotic compactness of the process (see [11, Definitions 2.3, 2.4]), which implies nonemptiness, compactness and pullback attraction for sets forming the $\omega-$limit family $\{\omega_W(\hat{D},t)\colon t\in\mathbb{R}\}$ for $\hat{D}\in\mathcal{D}$ ([11, Proposition 2.6]). In particular, in reflexive and strictly convex Banach spaces the asymptotic compactness of the process is also equivalent ([11, Propositions 2.8, 2.10]) to the flattening condition (cf. [11, Definition 2.7]), used in applications to verify the asymptotic compactness. Next, I use the asymptotic compactness together with the asymptotic closedness of the process ([11, Definition 2.11]) and the pullback dissipativity ([11, Definition 2.13]) to show invariance of $\omega-$limit families in [11, Proposition 2.15]. In the main theorem of the paper [11, Theorem 2.16] I show that if, additionally, the process is closed, then there exists a bi-space pullback global $\mathcal{D}-$attractor

$$\mathcal{A}(t)=\mathrm{cl}_W\bigcup_{\hat{D}\in\mathcal{D}}\omega_W(\hat{D},t)\subset\omega_W(\hat{B}_0,t),\ t\in\mathbb{R},$$

where $\hat{B}_0$ is the family absorbing $\mathcal{D}$ in the pullback sense. In the case of $\hat{B}_0\in\mathcal{D}$ the assumption on the closedness of the process can be weakened by supposing just the asymptotic closedness (cf. [11, Corollary 2.17]). Then we have $\mathcal{A}(t)=\omega_W(\hat{B}_0,t)$, $t\in\mathbb{R}$. The theory is illustrated by the Dirichlet problem for the nonautonomous reaction-diffusion equation

$$\begin{cases} u_t=\Delta u-f(u)+g(t),\ t>s,\ x\in\Omega\subset\mathbb{R}^N \text{ bounded },\\ u(t,x)=0,\ t>s,\ x\in\partial\Omega,\quad u(s,x)=u_0(x),\ x\in\Omega,\end{cases}$$

presented in the paper [ŁU], for which there exists the $(L^2(\Omega)-H_0^1(\Omega))$ pullback global $\mathcal{D}-$attractor with the universe

$$\mathcal{D}=\{\hat{D}=\{D(t)\subset L^2(\Omega)\colon t\in\mathbb{R}\}\colon \lim_{s\to-\infty}e^{\lambda_1 s}\sup\{\|u\|^2_{L^2(\Omega)}:u\in D(s)\}=0\},$$

which is stated in [11, Theorem 3.4].

In the paper [12], written jointly with Everaldo Bonotto, Matheus Bortolan and Rodolfo Collegari, we consider the problem of the influence of small perturbations of impulsive dynamical systems on their asymptotic behavior described with the aid of precompact global attractors in the paper [4] presented in the part devoted to the scientific achievement. Although the remarkable feature of impulsive semigroups $\{\tilde{\pi}_\eta(t)\colon t\geqslant 0\}$ is their discontinuity, the study of continuous dependence of precompact global attractors in the sense of Hausdorff semidistance is not meaningless if the elements generating the impulsive dynamical system depend continuously on the parameter. To this end, we consider a family of impulsive dynamical systems $\{(X,\pi_\eta,M_\eta,I_\eta)\}_{\eta\in[0,1]}$ and assume the continuity at $\eta=0$ of the continuous semigroups $\{\pi_\eta(t)\colon t\geqslant 0\}$ uniformly on compact subsets of $[0,\infty)\times X$, the continuity at $\eta=0$ of the impulsive sets M_η w.r.t. the Hausdorff distance, joint continuity at zero of the impulsive functions $I_\eta\colon M_\eta\to X$ and suppose that the impulsive sets and their images by the impulsive functions are disjoint for small perturbations. Moreover, we define collective tube conditions (cf. [12, Definitions 3.3, 3.4]), being an extension of the conditions known from the papers [CI2, 4], which describe the behavior of the semigroups π_η near the impulsive sets M_η with respect to the parameter η. On account of these assumptions we show in [12, Theorem 3.12] the joint continuity at zero (on $X \setminus M_0$) of the function ϕ_η of the smallest positive time to reach M_η (cf. (33)), whereas in [12, Corollary 3.17] we prove the joint continuity with correction at $\eta = 0$ (on $X \setminus M_0$) of the impulsive semigroups $\tilde{\pi}_\eta$. Both results are fundamental for the proof of the main theorem [12, Theorem 4.2] on the upper semicontinuity at $\eta = 0$ of the precompact global attractors $\mathcal{A}_\eta$ for the family of impulsive dynamical systems $\{(X, \pi_\eta, M_\eta, I_\eta)\}_{\eta\in[0,1]}$ (see (35)). We apply this theorem to a simple planar system with a perturbation and with given impulsive functions. The last part of the paper contains the theorem ([12, Theorem 6.3]) on the lower semicontinuity at $\eta = 0$ of the family of precompact global attractors for a particular family of impulsive dynamical systems.

In the paper [13], written jointly with Pedro Marín-Rubio, we study the nonautonomous semilinear parabolic equation with dynamical boundary condition of the form

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \kappa u + f_1(u) = h_1(t) & \text{in } \Omega \times (s, \infty), \\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial \vec{n}} + f_2(u) = h_2(t) & \text{on } \partial\Omega \times (s, \infty), \\ u(x, s) = u_s(x) & \text{for } x \in \Omega, \\ u(x, s) = \varphi_s(x) & \text{for } x \in \partial\Omega, \end{cases} \tag{43}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, with the Lipschitz boundary $\partial\Omega$ and $\kappa > 0$. Our aim was to prove for (43) the existence of a pullback exponential attractor and to obtain an estimate of the finite fractal dimension of the pullback global attractor, whose existence in $H = L^2(\Omega) \times L^2(\partial\Omega)$, under appropriate assumptions on the nonlinearity, was earlier proved in the paper [A-M-R1] and its regularity investigated in the paper [A-M-R2].

We assume that the functions $f_1, f_2 \in C(\mathbb{R})$ are almost monotonic, i.e., the functions $f_i(u) + lu$ are nondecreasing with some $l > 0$, satisfy the growth condition

$$|f_i(u) - f_i(v)| \leqslant L\,|u - v| \left(1 + |u|^{p_i-2} + |v|^{p_i-2}\right), \ u, v \in \mathbb{R}, \ i = 1, 2, \tag{44}$$

and the dissipativity condition

$$f_i(u)u \geqslant \alpha\,|u|^{p_i} - \beta, \ u \in \mathbb{R}, \ i = 1, 2, \tag{45}$$

with some constants $p_i \geqslant 2$, $\alpha, L > 0$, $\beta \geqslant 0$, whereas $\vec{h} = (h_1, h_2) \in L^2_{loc}(\mathbb{R}; H)$.

In [13, Theorem 2.2] we show the existence of global weak solutions of (43) for initial conditions $(u_s, \varphi_s) \in L^2(\Omega) \times L^2(\partial\Omega)$ using the classical approach of J.-L. Lions ([LI]). Next, applying the results of my paper [3], we prove in [13, Theorem 4.5] the existence of a pullback exponential attractor for (43) in the space H provided that the nonautonomous term $\vec{h}$ is translation bounded, i.e.,

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \left|\vec{h}(\tau)\right|_H^2 d\tau \leqslant K,$$

and the nonlinearities f_i, $i = 1, 2$, satisfy (44) and (45) with suitable exponents p_i (see [13, (4.5)]). If the boundary $\partial\Omega$ is sufficiently smooth and f_1, f_2 satisfy additionally

$$|f_1(s) - f_2(s)| \leqslant C(1 + |s|), \ s \in \mathbb{R},$$

then the requirements for the exponents $p_1 = p_2 = p$ can be weakened (cf. [13, (4.15)]). In particular, for $N = 2$ the nonlinearities $f_i(u) = u^3 - u$, $u \in \mathbb{R}$, as well as any polynomials of odd degree with positive leading coefficient are allowed. This also shows that the regular pullback global attractor studied in [A-M-R2] has a uniform bound of the fractal dimension if the term $\vec{h}$ is translation bounded.

In the paper [13] we also consider the Lipschitz case ($p_1 = p_2 = 2$) and prove in [13, Theorem 5.4] the existence of a pullback exponential attractor for (43) in H even if the nonautonomous term grows exponentially in the past and in the future, i.e., if $\vec{h}$ satisfies

$$\left|\vec{h}(t)\right|_H^2 \leqslant K e^{\theta|t|},\ t \in \mathbb{R},$$

with some $K > 0$ and $0 \leqslant \theta < 2(\lambda_1 + \alpha)$, where $\lambda_1 > 0$ is the first eigenvalue of the linear operator A_0 from the abstract Cauchy problem corresponding to the system (43).

6. Paper to be published

[I] Radosław Czaja, Waldyr M. Oliva, Carlos Rocha, On a definition of Morse-Smale evolution processes, paper in review.

In the years 2009-2014 I stayed in the Instituto Superior Técnico in Lisbon and participated in the research carried out there by Waldyr M. Oliva and Carlos Rocha. This research concentrated around an appropriate generalization of the notion of the Morse-Smale semigroup to the nonautonomous case. In particular, the definition of the Morse-Smale process should imply the property of openness, i.e., a small nonautonomous perturbation of an autonomous equation generating a Morse-Smale semigroup is a Morse-Smale process. In the meantime appeared the paper [B-C-L], in which the authors formulate a definition of the Morse-Smale process (cf. [B-C-L, Definition 2.18]) if the process comes from a gradient system without (hyperbolic) periodic orbits and containing only hyperbolic stationary solutions. We judged this as an oversimplification and we concentrated on the study of nonautonomous perturbations of periodic orbits. We constructed an example of an autonomous system of ordinary differential equations with a hyperbolic periodic orbit for which small appropriate nonautonomous perturbations preserve the corresponding isolated invariant cylinder though the dynamics on it changes diametrically. Using the isolated invariant manifolds (cf. integral manifolds considered e.g. by J.K. Hale [HA3, Theorem VII.7.1] and D. Henry [HE1, Theorem 9.1.1]) we showed their persistence under the perturbation of hyperbolic periodic orbits. Later we formulated the definition of the Morse-Smale process substituting the nonwandering points from the classical definition by an appropriate recurrent behavior for the evolution process. This allowed to prove the above-mentioned property of openness of Morse-Smale processes for semilinear parabolic equations. These results have been included in the manuscript [I] and are currently under review.

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