AUTHOR'S REVIEW OF HIS RESEARCH, ACHIEVEMENTS AND PUBLICATIONS

1. Name: **Szymon Żeberski**
2. Obtained diplomas and academic degrees:
   - M.Sc. in Mathematics, Mathematical Institute, University of Wrocław, 2000
   - Ph.D. in Mathematics, Mathematical Institute, University of Wrocław, 2004
3. Employment in scientific institutions:
   - Assistant Lecturer, Mathematical Institute, University of Wrocław, 2004–2005,
   - Assistant Professor, Mathematical Institute, University of Wrocław, 2005–2007,
   - Assistant Professor, Institute of Mathematics and Computer Science, Wrocław University of Technology, 2007–2014,
   - Assistant Professor, Department of Computer Science, Faculty of Fundamental Problems of Technology, Wrocław University of Technology, since 2014.
4. Achievements in the form of series of publications under the title

   Nonmeasurability in Polish spaces

   with respect to Article 16 Paragraph 2 of the Act of 14 March 2003 on Academic Degrees and Title and on Degrees and Title in the Field of Art.

LIST OF PUBLICATIONS INCLUDED IN THE ACHIEVEMENT MENTIONED ABOVE

[H1] Sz. Żeberski, *On weakly measurable functions*, BULLETIN OF THE POLISH ACADEMY OF SCIENCES, MATHEMATICS, 53 (2005), 421-428,

[H2] Sz. Żeberski, *On completely nonmeasurable unions*, MATHEMATICAL LOGIC QUARTERLY, 53 (1) (2007), 38-42,

[H3] R. Rałowski, Sz. Żeberski, *Complete nonmeasurability in regular families*, HOUSTON JOURNAL OF MATHEMATICS, 34 (3) (2008), 773-780,

[H4] R. Rałowski, Sz. Żeberski, *On nonmeasurable images*, CZECHOSLOVAK MATHEMATICAL JOURNAL, 60 (135) (2010), 424-434,

[H5] R. Rałowski, Sz. Żeberski, *Completely nonmeasurable unions*, CENTRAL EUROPEAN JOURNAL OF MATHEMATICS, 8 (4) (2010), 683-687,

[H6] Sz. Żeberski, *Inscribing nonmeasurable sets*, ARCHIVES FOR MATHEMATICAL LOGIC, 50 (3) (2011), 423-430,

[H7] R. Rałowski, Sz. Żeberski, *Generalized Luzin sets*, HOUSTON JOURNAL OF MATHEMATICS, 39 (3) (2013), 983-993,

[H8] M. Michalski, Sz. Żeberski, *Some properties of I-Luzin sets*, TOPOLOGY AND ITS APPLICATIONS, 189 (2015), 122-135.

The discussion of the results of the above-mentioned publications

History of the theme and motivation of research

We will use standard set-theoretic notation and terminology.

For fixed uncountable Polish space T by $\mathcal{M}$ we will denote a σ-ideal of meager subsets of T. If there is a measure given in T then by $\mathcal{N}$ we will denote a σ-ideal of null sets.

One of the first results concerning nonmeasurable unions of subsets of the real line is the following result of Kuratowski from [Ku].

Theorem 1 (Kuratowski)**.** *Assume CH. For every family $\mathcal{A} \subseteq \mathcal{M}$ of pairwise disjoint meager sets such that $\bigcup \mathcal{A} \notin \mathcal{M}$ there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ with a union $\bigcup \mathcal{A}'$ without Baire property.*

The conclusion of the above theorem remains true in ZFC. It was proved by L. Bukovsky in [Bu].

Theorem 2 (Bukovsky)**.** (1) *For every partition $\mathcal{A} \subseteq \mathcal{M}$ of the real line into meager sets, there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ with a union $\bigcup \mathcal{A}'$ without Baire property.*

(2) *For every partition $\mathcal{A} \subseteq \mathcal{N}$ of the real line into null sets, there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ with Lebesgue nonmeasurable union $\bigcup \mathcal{A}'$.*

Unlike the proof of Kuratowski's theorem, the proof given by Bukovsky uses a nonelementary method of generic ultrapower applied to the Cohen forcing in the case of category and applied to the Solovay forcing in the case of measure.

A generalization of the result obtained by Bukovsky was given by J. Brzuchowski, J. Cichoń, E. Grzegorek and Cz. Ryll-Nardzewski in [BCGR].

Let us recall that if $\mathcal{I} \subseteq P(T)$ is a σ-ideal, then $\mathcal{A}$ is called a *base* of $\mathcal{I}$ if $\mathcal{A} \subseteq \mathcal{I}$ and $(\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq I)$. We say that $\mathcal{I}$ has *Borel base* if there exists a base of $\mathcal{I}$ consisting of Borel sets. We define analytic, co-analytic and BP-base analogously. In the latter case we require that there exists a base consisting of sets with Baire property.

Theorem 3 (Brzuchowski, Cichoń, Grzegorek, Ryll-Nardzewski)**.** *If $\mathcal{I}$ is a σ-ideal on a Polish space T with Borel base, containing singletons, then for every point-finite family $\mathcal{A} \subseteq \mathcal{I}$ (i.e. such that such that $\forall x \in T$ $\{A \in \mathcal{A} : x \in A\}$ is finite) such that $\bigcup \mathcal{A} \notin \mathcal{I}$ there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ which is not $\mathcal{I}$-measurable i.e. does not belong to the σ-field generated by σ-ideal $\mathcal{I}$ and σ-field of all Borel subsets* $\mathrm{Bor}(T)$*.*

The proof of the above theorem is elementary: it uses elements of classical descriptive set theory.

This theorem cannot be extended to families $\mathcal{A} \subseteq \mathcal{N}$ which are point-countable i.e. such that

$$(\forall x \in T)\ (\{A \in \mathcal{A} :\ x \in A\} \text{ is countable}).$$

This was proved by D. Fremlin in [Fr]. He considered the model obtained from a model of GCH by adding ω_2 independent Cohen reals. He constructed there a point-countable cover $\mathcal{A} \subseteq \mathcal{N}$ of reals by null sets such that for every subfamily $\mathcal{A}' \subseteq \mathcal{A}$ its union $\bigcup \mathcal{A}'$ is measurable in the sense of Lebesgue. A similar result remains true in the case of meager sets. In this case it is enough to consider a model obtained from the model of GCH by adding ω_2 independent Solovay reals.

Definition 1. *Let $\mathcal{I}$ be a σ-ideal of subsets of an uncountable Polish space T. Assume that $\mathcal{I}$ has Borel base. A set $A \subseteq T$ is called*

(1) *$\mathcal{I}$-measurable, if A belongs to the σ-field generated by $Bor(T)$ Borel subsets of T and the σ-ideal $\mathcal{I}$,*
(2) *$\mathcal{I}$-nonmeasurable, if A is not $\mathcal{I}$-measurable,*
(3) *completely $\mathcal{I}$-nonmeasurable, if for every Borel set $B \notin \mathcal{I}$ the intersection $A \cap B$ is $\mathcal{I}$-nonmeasurable.*

If $\mathcal{I}$ is the σ-ideal of countable sets, A is $\mathcal{I}$-nonmeasurable if and only if A is not a Borel set, and A is completely $\mathcal{I}$-nonmeasurable if and only if A is a Bernstein set.

If $\mathcal{I} = \mathcal{N}$ is the σ-ideal of Lebesgue null subsets of the space $T = [0, 1]$, a set A is completely $\mathcal{I}$-nonmeasurable if and only if its inner measure is equal to 0 and its outer measure is equal to 1.

One of the directions of research on nonmeasurable unions is verifying the following hypothesis:

Hypothesis 1. *Let $\mathcal{I}$ be a σ-ideal of subsets of an uncountable Polish space T. Assume that $\mathcal{I}$ has Borel base. Let $\mathcal{A}$ be a point-finite cover of T by subsets from $\mathcal{I}$. Then there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ with completely $\mathcal{I}$-nonmeasurable union $\bigcup \mathcal{A}'$.*

In the case $\mathcal{I} = \mathcal{N}$ it is still an open question whether for every partition of the unit interval $[0, 1]$ into null set there exists a subfamily whose union has inner measure 0 and outer measure 1. Partial result were obtained by D. Fremlin and S. Todorcević in [FT].

Theorem 4 (Fremlin, Todorcević)**.** *For every partition $\mathcal{P}$ of the unit interval $[0, 1]$ into null sets and for every $\varepsilon > 0$ we can find a subfamily $\mathcal{A} \subseteq \mathcal{P}$ whose union $\bigcup \mathcal{A}$ has inner measure smaller than ε and outer measure greater than $1 - \varepsilon$.*

The following result of M. Gitik and S. Shelah from [GS2] provides the inspiration for further potential generalizations.

Theorem 5 (Gitik, Shelah)**.** *Let $(A_n)_{n\in\omega}$ be a sequence of subsets of the real line $\mathbb{R}$. Then there exists a sequence $(B_n)_{n\in\omega}$ satisfying the following conditions*
(1) *$B_n \subseteq A_n$, for each n,*
(2) *if $n \neq m$ then $B_n \cap B_m = \emptyset$,*
(3) *$\lambda^*(A_n) = \lambda^*(B_n)$, for every n,*

where λ^* denotes the outer Lebesgue measure.

Proof of the above result uses the method of generic ultrapower. It is based on showing that the Boolean algebra of the form $P(\kappa)/I$ cannot be isomorphic to *Cohen* $\times$ *Random* (or *Random* $\times$ *Cohen*).

Let us notice that Theorem 5 is a natural generalization of the classical result of Luzin that every subset A of reals can be partitioned into two subsets of the same outer measure as the outer measure of A.

In [CK] J. Cichoń and A. Kharazishvili showed a series of applications of theorems about the existence of a subfamily with nonmeasurable union.

Let $\mathcal{X}$ be any topological space. A function $f : T \to \mathcal{X}$ is called *$Bor[\mathcal{I}]$-measurable*, if for every open set U the preimage $f^{-1}[U]$ belongs to the σ-field generated by Borel subsets of T and the σ-ideal $\mathcal{I}$.

Theorem 6 (Cichoń, Kharazishvili)**.** *Let E be a metric space and $f : T \to E$ be a $Bor[\mathcal{I}]$-measurable function. Then there exists a set $A \in \mathcal{I}$ such that $f[T \setminus A]$ is a separable space.*

A natural corollary of the above theorem (for $\mathcal{I} = \{\emptyset\}$) is the classical Frolik theorem.

Theorem 7 (Frolik)**.** *Let E be a metric space and $f : T \to E$ be a Borel function. Then $rng(f)$ is a separable space.*

Theorem 8 (Cichoń, Kharazishvili)**.** *Let E be a metric group and $f, g : T \to E$ be a $Bor[\mathcal{I}]$-measurable functions. Then $f + g$ is a $Bor[\mathcal{I}]$-measurable function.*

A particular case of unions of sets are algebraic sums defined for an abelian group $(G, +)$. Let $A, B \in P(G)$ be any subsets of given group. An algebraic sum $A + B$ is defined by the following formula:

$$A + B = \{a + b \in G : \ (a, b) \in A \times B\}.$$

W. Sierpiński in [Si1] proved that there exist subsets X, Y of the real line $\mathbb{R}$ such that the algebraic sum $X + Y$ is not Lebesgue measurable.

A result of Sierpiński was strengthened by M. Kysiak in [Ky]. He showed the following result.

Theorem 9 (Kysiak)**.** *Assume that a σ-ideal $\mathcal{I}$ of subsets of reals contains all singletons and a pair $(\mathcal{I}, \mathcal{A})$ has perfect set property. (A pair $(\mathcal{I}, \mathcal{A})$ has perfect set property if every set $B \in \mathcal{A} \setminus \mathcal{I}$ contains nonempty perfect set.) Then for every subset $A \subseteq \mathbb{R}$ such that $A + A \notin \mathcal{I}$ there exists $X \subseteq A$ such that $X + X \notin \mathcal{A}$.*

$(\mathcal{N}, \mathcal{LM})$ and $(\mathcal{M}, \mathcal{BP})$ are examples of pairs possessing perfect set property (here $\mathcal{LM}$ denotes the σ-algebra of all Lebesgue measurable sets, $\mathcal{BP}$ denotes the family of all subsets of $\mathbb{R}$ that have the Baire property). As a corollary we obtain the theorem of Ciesielski, Fejzic, Freiling from [CFF] saying that if $A \subseteq \mathbb{R}$ and $A + A$ has positive outer measure then there exists a set $X \subseteq A$ such that $X + X$ is Lebesgue nonmeasurable. The perfect set property of the pair $(\mathcal{M}, \mathcal{BP})$ implies the analogous theorem for the σ-ideal of meager sets $\mathcal{M}$. In [CFF] authors proved also the following result: if $A \subseteq \mathbb{R}$ is such that $A + A \notin \mathcal{N}$ then there exists a null subset $X \subseteq A$ such that $X + X$ is nonmeasurable. The analogous result is true in the case of category.

In the research concerning families of subsets of Polish space, in particular in the research concerning σ-ideals, an important role is played by cardinal coefficients. For a given family $\mathcal{F} \subseteq P(X)$ of subsets of a given Polish space T we define the cardinal coefficients in the following way:

$$\begin{aligned}
&\mathrm{add}(\mathcal{F}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F} \ \wedge \ \bigcup \mathcal{A} \notin \mathcal{F}\},\\
&\mathrm{non}(\mathcal{F}) = \min\{|A| : A \subseteq X \ \wedge \ A \notin \mathcal{F}\},\\
&\mathrm{cov}(\mathcal{F}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F} \wedge \ \bigcup \mathcal{A} = T\},\\
&\mathrm{cov}_h(\mathcal{F}) = \min\{|\mathcal{A}| : \ \mathcal{A} \subseteq \mathcal{F} \wedge (\exists B \in Bor(T) \setminus \mathcal{F}) \ B \subseteq \bigcup \mathcal{A}\},\\
&\mathrm{cof}(\mathcal{F}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F} \ \wedge \ (\forall B \in \mathcal{F})(\exists A \in \mathcal{A}) \ \ B \subseteq A\}.
\end{aligned}$$

Moreover the bounding number and the dominating number

$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \omega^\omega \wedge (\forall x \in \omega^\omega)(\exists y \in \mathcal{B}) \ \neg(y \leq^* x)\}$$

$$\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \subseteq \omega^\omega \wedge (\forall x \in \omega^\omega)(\exists y \in \mathcal{D}) \ x \leq^* y\}$$

(where $f \leq^* g$ denotes that $(\exists m \in \omega)(\forall n \geq m) f(n) \leq g(n)$) are also connected to cardinal coefficients for the σ-ideal of meager sets and this connection is described in so called *Cichoń's* *diagram* (see e.g. [BJS]).

$$\begin{array}{ccccccccc}
& & \mathrm{cov}(\mathcal{N}) & \longrightarrow & \mathrm{non}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{N}) & \longrightarrow & \mathfrak{c} \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
& & | & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & | & & \\
& & | & & \uparrow & & \uparrow & & | & & \\
\omega_1 & \longrightarrow & \mathrm{add}(\mathcal{N}) & \longrightarrow & \mathrm{add}(\mathcal{M}) & \longrightarrow & \mathrm{cov}(\mathcal{M}) & \longrightarrow & \mathrm{non}(\mathcal{N}) & &
\end{array}$$

Luzin sets and Sierpiński sets also occur in the research concerning nonmeasurable sets. They were defined and studied in [Lu], [Si2].

Definition 2. *A set $A \subseteq T$ is called*

(1) *a Luzin set, if A is uncountable and the intersection $A \cap M$ is countable for every meager set $M \in \mathcal{M}$,*
(2) *a Sierpiński set, if A is uncountable and the intersection $A \cap N$ is countable for every null set $N \in \mathcal{N}$.*

The existence of a Luzin set (a Sierpiński set) is independent of ZFC theory. Under CH, above mentioned sets exist, but under $MA + \neg CH$ they do not exist.

In [Sch] M. Scheepers gave a characterization of Luzin's set in terms of covering properties (similar to the Rothberger property).

The notion of Luzin set can be naturally generalized replacing the ideal of meager sets $\mathcal{M}$ by any other ideal of subsets of space T.

In [BH] T. Bartoszyński and L. Halbeisen consider $\mathcal{K}$-Luzin sets, where $\mathcal{K}$ is a σ-ideal generated by compact subsets of the Baire space ω^ω. Authors show that existence of a $\mathcal{K}$-Luzin set of cardinality $\mathfrak{c}$ is equivalent to Banach-Kuratowski theorem about matrix of subsets of $[0, 1]$.

Generalized Luzin sets were also considered. Let us recall that $A \subseteq T$ is a *generalized Luzin set*, if for every meager set $M \in \mathcal{M}$ we have $|M \cap A| < |A|$. Generalized Luzin sets were studied e.g. by Bukovsky in [Buk]. Also in that case, the ideal of meager sets $\mathcal{M}$ can be replaced by another ideal $\mathcal{I}$. In that way we obtain the so called *generalized $\mathcal{I}$-Luzin sets.*

The discussion of the results

Unions of sets. We will present results that are partial solutions of the following problem: does every point-finite family of sets from a σ-ideal $\mathcal{I}$ contain a subfamily with completely $\mathcal{I}$-nonmeasurable union? We will focus on σ-ideals $\mathcal{I}$ possessing *Suslin property* (known also as c.c.c. ideals), i.e. such σ-ideals $\mathcal{I}$ that every family of pairwise disjoint Borel sets outside $\mathcal{I}$ is countable. Examples of such σ-ideals are $\mathcal{N}$ and $\mathcal{M}$. Each σ-ideal with Suslin property has also *hull property* i.e. for every subset X of a space T there exists the smallest (modulo a set from ideal $\mathcal{I}$) $\mathcal{I}$-measurable set containing X (modulo a set from ideal $\mathcal{I}$). By $[X]_{\mathcal{I}}$ we denote the member of Boolean algebra $\mathcal{B}[\mathcal{I}]/\mathcal{I}$ representing a class of all minimal $\mathcal{I}$-measurable subsets of T containing X modulo a set from $\mathcal{I}$.

We will assume additionally that some large cardinals smaller than continuum do not exist.

Let us recall the definition given by D. Fremlin (see [Fre]).

Definition 3. *An uncountable regular cardinal number κ is called quasi-measurable if there exists a σ-ideal $\mathcal{J} \subseteq P(\kappa)$ such that the Boolen algebra $P(\kappa)/\mathcal{J}$ satisfies c.c.c., i.e. every collection of pairwise disjoint elements of this algebra is countable.*

Let us notice that if κ is a measurable cardinal or a real-valued measurable cardinal, then κ is quasi-measurable. On the other hand, the Ulam matrix guaranties that every quasi-measurable cardinal is weakly inaccessible. The quasi-measurable cardinals are so called large cardinals.

Let us give a natural generalization of the notion of complete $\mathcal{I}$-nonmeasurability which additionally depends on a given subset X of a space T.

Definition 4. *Let T be an uncountable Polish space, $\mathcal{I}$ – a σ-ideal of subsets of T having Borel base. Fix $X \notin \mathcal{I}$. We say that $A \subseteq X$ is completely $\mathcal{I}$-nonmeasurable in X if*

$$(\forall B \in Bor \setminus \mathcal{I})(B \cap X \notin \mathcal{I} \Rightarrow (A \cap B \notin \mathcal{I}) \wedge (A \cap (X \setminus B) \notin \mathcal{I})).$$

Theorem 10 ([H2], Theorem 3.6)**.** *Assume that there is no quasi-measurable cardinal $\kappa \leq \mathfrak{c}$. Let T be an uncountable Polish space, $\mathcal{I}$ – a σ-ideal of subsets of T possessing Borel base and having Suslin property. Then for every point-finite family $\mathcal{A} \subseteq \mathcal{I}$ with union outside $\mathcal{I}$ we can find a subfamily $\mathcal{A}'$ such that $\bigcup \mathcal{A}'$ is completely $\mathcal{I}$-nonmeasurable in $\bigcup \mathcal{A}$.*

The proof of the latter theorem is based on the following three lemmas (the assumptions in these lemmas are the same as in Theorem 10).

Lemma 1 ([H2], Theorem 3.3)**.** *Let $\{A_\xi : \xi \in \omega_1\}$ be any family of subsets of T. Then there exists a family $\{I_\alpha : \alpha \in \omega_1\}$ of pairwise disjoint subsets of ω_1 such that for every $\alpha, \beta \in \omega_1$ we have $\left[\bigcup_{\xi \in I_\alpha} A_\xi\right]_{\mathcal{I}} = \left[\bigcup_{\xi \in I_\beta} A_\xi\right]_{\mathcal{I}}$.*

Lemma 2 ([H2], Lemma 3.4)**.** *There exist families $\mathcal{A}_\alpha \subseteq \mathcal{A}$ for $\alpha \in \omega_1$ satisfying the following conditions*

(1) $\bigcup \mathcal{A}_\alpha \notin \mathcal{I}$,
(2) $\alpha \neq \beta$ *implies* $\mathcal{A}_\alpha \cap \mathcal{A}_\beta = \emptyset$,
(3) $[\bigcup \mathcal{A}_\alpha]_{\mathcal{I}} = [\bigcup \mathcal{A}_\beta]_{\mathcal{I}}$.

Lemma 3 ([H2], Lemma 3.5)**.** *Let $\mathcal{C}$ be any point-finite family of subsets of space T. Then there exists a subfamily $\mathcal{C}' \subseteq \mathcal{C}$ such that a set $\mathcal{C} \setminus \mathcal{C}'$ is countable and*

$$(\forall B \in \mathcal{B} \setminus \mathcal{I})(\forall C \in \mathcal{C}')(B \cap \bigcup \mathcal{C} \notin \mathcal{I} \Rightarrow \neg(B \cap \bigcup \mathcal{C} \subseteq B \cap C)).$$

Theorem 10 was strengthened in [H5] (written jointly with R. Rałowski).

Theorem 11 ([H5], Theorem 1.3)**.** *Assume that there is no quasi-measurable cardinal $\kappa < \mathfrak{c}$. Let T be an uncountable Polish space, $\mathcal{I}$ – a σ-ideal of subsets of T possessing Borel base and having Suslin property. Then for every point-finite cover $\mathcal{A} \subseteq \mathcal{I}$ of the space T we can find a subfamily $\mathcal{A}'$ such that $\bigcup \mathcal{A}'$ is completely $\mathcal{I}$-nonmeasurable.*

The proof of this theorem uses Lemmas 1, 2, 3 and two more additional theorems.

Theorem 12 ([H5], Theorem 2.1)**.** *Assume that $\mathcal{A} \subseteq \mathcal{I}$ is a cover of a Polish space T such that for every set $D \subseteq T$ of cardinality smaller than $\mathfrak{c}$ a union $\bigcup_{x \in D} \bigcup \mathcal{A}(x)$ does not contain any Borel set outside $\mathcal{I}$. (A set $\mathcal{A}(x)$ denotes the collection of all elements of a family $\mathcal{A}$ which contain x, i.e. $\mathcal{A}(x) = \{A \in \mathcal{A} : x \in A\}$.) Then $\mathcal{A}$ contains pairwise disjoint families $\mathcal{A}_\xi$ for $\xi < \mathfrak{c}$ such that $\bigcup \mathcal{A}_\xi$ is a completely $\mathcal{I}$-nonmeasurable set.*

Theorem 13 ([H5], Theorem 2.2). *Assume that there is no quasi-measurable cardinal $\kappa < \mathfrak{c}$. Let $\mathcal{A} \subseteq \mathcal{I}$ be a family such that for every x the set $\mathcal{A}(x)$ has cardinality smaller than $\mathfrak{c}$. If $\bigcup \mathcal{A} \notin \mathcal{I}$ then we can find an uncountable collection of subfamilies $\mathcal{A}_\alpha \subseteq \mathcal{A}$, $\alpha \in \omega_1$ such that $\bigcup \mathcal{A}_\alpha \notin \mathcal{I}$ and $\bigcup \mathcal{A}_\alpha \cap \bigcup \mathcal{A}_\beta \in \mathcal{I}$ for $\alpha \neq \beta$.*

Let us now concern generalizations of Theorem 5 about inscribing nonmeasurable sets into given sets (families of sets).

Theorem 14 ([H6], Theorem 3.2). *Let $\mathcal{A}$ be a family of pairwise disjoint subsets of space T consisting of first category sets. Assume that $\bigcup \mathcal{A} \notin \mathcal{M}$. For $n \in \omega$ let us fix any $\mathcal{A}_n \subseteq \mathcal{A}$. Then there exists $\mathcal{B} \subseteq \mathcal{A}$ satisfying the following conditions:*

(1) $\bigcup \mathcal{B} \notin \mathcal{M}$,
(2) *for every $n \in \omega$ $\bigcup \mathcal{B} \cap \bigcup \mathcal{A}_n$ does not contain any $\mathcal{M}$-positive Borel set modulo $\bigcup \mathcal{A}_n$, i.e.*

$$(\forall n)(\neg \exists U)(U \textit{ is open } \wedge U \cap \bigcup \mathcal{A}_n \notin \mathcal{M} \wedge U \cap \bigcup \mathcal{A}_n \setminus \bigcup \mathcal{B} \in \mathcal{M}).$$

The proof of this theorem consists in obtaining a contradiction with the existence of a family that does not fulfil the conclusion of the theorem. Namely, using such family we construct a Boolean algebra of the form $P(\kappa)/\mathcal{I}$. Then we show that this algebra is atomless and contains a countable dense subset. It means that it is the only (up to isomorphism) such algebra, i.e. Cohen algebra. By the result of Gitik and Shelah (see [GS1]) it is impossible.

Theorem 15 ([H6], Theorem 3.6). *Let $\mathcal{A} \subseteq \mathcal{M}$ be a family of pairwise disjoint subsets of space T. Assume that $\bigcup \mathcal{A} \notin \mathcal{M}$. For $n \in \omega$ let us fix any $\mathcal{A}_n \subseteq \mathcal{A}$. Then there exists $\mathcal{B} \subseteq \mathcal{A}$ satisfying the following conditions:*

(1) $[\bigcup \mathcal{B}]_{\mathcal{M}} = [\bigcup \mathcal{A}]_{\mathcal{M}}$,
(2) *for every $n \in \omega$ a set $\bigcup \mathcal{B} \cap \bigcup \mathcal{A}_n$ does not contain any $\mathcal{M}$-positive Borel set modulo $\bigcup \mathcal{A}_n$.*

The proof of the latter theorem uses Theorem 14. In the first step we divide the space $\bigcup \mathcal{A}$ into two pieces. The first one contains fragments on which Boolean algebras fulfils Suslin property. The second subspace does not have this property hereditarily. First, we make a construction in the second subspace. We use Erdős-Alaoglu theorem, or, more precisely, its version following from the proof from Taylor's paper [Ta]. Then we deal with "c.c.c. subspace" and use Theorem 14 to build a family by transfinite induction. Suslin property ensures that the construction stops at step $< \omega_1$.

The next theorem is a variant of Theorem 5 from [GS2] and, at the same time, a generalization of Theorem 52 from [P3] about the existence of a subfamily with completely $\mathcal{M}$-nonmeasurable union in a given partition into meager sets.

Theorem 16 ([H6] Theorem 3.7). *Assume that $\mathcal{A} \subseteq \mathcal{M}$ is a partition of a Polish space T. Let $\mathcal{A}_n \subseteq \mathcal{A}$ for $n \in \omega$. Then there exists $\mathcal{B}_n \subseteq \mathcal{A}_n$ such that*

(1) $\mathcal{B}_n \cap \mathcal{B}_m = \emptyset$ *for $n \neq m$,*
(2) $\mathcal{B}_n \subseteq \mathcal{A}_n$,
(3) $[\bigcup \mathcal{A}_n]_{\mathcal{M}} = [\bigcup \mathcal{B}_n]_{\mathcal{M}}$.

Now, let us focus on inscribing subfamilies in the case of any σ-ideal with Suslin property. Let us start with the following lemma.

Theorem 17 ([H6], Lemma 4.3). *Assume that there is no quasi-measurable cardinal smaller than continuum. Let $\mathcal{A} \subseteq \mathcal{I}$ be a point-finite family. Let $(\mathcal{A}_n : n \in \omega)$ be a sequence of subsets of $\mathcal{A}$. Then we can find a sequence $(\mathcal{B}_n : n \in \omega)$ fulfilling the following conditions:*

(1) $\mathcal{B}_n \cap \mathcal{B}_m = \emptyset$ *for* $n \neq m$,
(2) $\mathcal{B}_n \subseteq \mathcal{A}_n$,
(3) $[\bigcup \mathcal{A}_n]_{\mathcal{I}} = [\bigcup \mathcal{B}_n]_{\mathcal{I}}$.

The latter theorem can be seen as a generalization of Theorem 10. The methods used in the proofs of both theorems are also similar.

We can strengthen the conclusion of the latter theorem inscribing uncountably many families into each family $\mathcal{A}_n$. More precisely, we have the following result.

Theorem 18 ([H6], Theorem 4.4). *Assume that there is no quasi-measurable cardinal smaller than continuum. Let $\mathcal{A} \subseteq \mathcal{I}$ be a point-finite family. Let $(\mathcal{A}_n : n \in \omega)$ be a sequence of subsets of $\mathcal{A}$. Then we can find a sequence $(\mathcal{B}_n^\xi : n \in \omega, \xi \in \omega_1)$ fulfilling the following conditions:*

(1) $\mathcal{B}_n^\xi \cap \mathcal{B}_m^\eta = \emptyset$ *for* $(n,\xi) \neq (m,\eta)$,
(2) $\mathcal{B}_n^\xi \subseteq \mathcal{A}_n$,
(3) $[\bigcup \mathcal{A}_n]_{\mathcal{I}} = [\bigcup \mathcal{B}_n^\xi]_{\mathcal{I}}$.

Let us now present results concerning nonmeasurable unions of families with some additional properties.

Let us start with the following definitions.

Definition 5. *Let $\mathcal{A}$ be an algebra of subsets of X. Let $F : X \to Y$ be a multifunction, i.e. $F \subseteq X \times Y$. We say that a multifunction F is $\mathcal{A}$-measurable if for every open subset U of the space Y the set*

$$F^{-1}[U] = \{x \in X :\ F_x \cap U \neq \emptyset\}$$

belongs to the family $\mathcal{A}$.

Definition 6. *Let $\mathcal{P}$ be a partition of a set X.*

(1) *A saturation of a set $A \subseteq X$ is a set $A^* = \bigcup\{E \in \mathcal{P} :\ E \cap A \neq \emptyset\}$.*
(2) *A partition $\mathcal{P}$ is called Borel measurable if a saturation of any open set is Borel.*
(3) *A partition $\mathcal{P}$ is called strongly Borel measurable if a saturation of any closed set is Borel.*

Let us notice that for second countable spaces strong Borel measurability implies Borel measurability of a partition.

Theorem 19 ([H3], Theorem 2.1). *Assume that any Borel set outside $\mathcal{I}$ contains a closed set outside $\mathcal{I}$. Let $\mathcal{A}$ be a strongly Borel measurable partition of a space T into closed subsets from $\mathcal{I}$. Then there is a subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}_0$ is completely $\mathcal{I}$-nonmeasurable.*

As a corollary we obtain the following result.

Corollary 1 ([H3], Corollary 3.2). *Let G be an abelian compact Polish group. Suppose that σ-ideal $\mathcal{I}$ is closed under translations. Assume that each Borel set outside $\mathcal{I}$ contains a closed set outside $\mathcal{I}$. Let $H < G$ be a perfect subgroup and $H \in \mathcal{I}$. Then there exists $T \subseteq G$ such that $T + H$ is completely $\mathcal{I}$–nonmeasurable in G.*

Theorem 20 ([H3], Theorem 2.2)**.** *Let $f : X \to Y$ be a Borel measurable function such that $f^{-1}(y) \in \mathcal{I}$ for every $y \in Y$. Then there is $T \subseteq Y$ such that $f^{-1}[T]$ is completely $\mathcal{I}$-nonmeasurable.*

Theorem 21 ([H3], Theorem 2.3)**.** *Assume that a σ-ideal $\mathcal{I}$ has Suslin property. Let $F : X \to Y$ be Borel measurable multifunction such that $F(x)$ is finite for every $x \in X$. Then there exists $T \subseteq Y$ such that $F^{-1}[T]$ is completely $\mathcal{I}$-nonmeasurable.*

The proof of this theorem uses Kuratowski–Ryll-Nardzewski selection theorem from [KR].

Theorem 22 ([H3], Theorem 2.4)**.** *Assume that a σ-ideal $\mathcal{I}$ has Suslin property. Let $F \subseteq X \times Y$ be an analytic set satisfying the following conditions:*

(1) $(\forall y \in Y)(F^y \in \mathcal{I})$,
(2) $X \setminus \pi[F] \in \mathcal{I}$, *where $\pi : X \times Y \to X$ is a projection on the first coordinate,*
(3) $(\forall x \in X)(|F_x| < \omega)$.

Then there exists a set $T \subseteq Y$ such that $F^{-1}[T]$ is completely $\mathcal{I}$-nonmeasurable.

Applications. In this subsection T denotes uncountable Polish space.

Definition 7. *Let $\mathcal{X}$ be a topological space. We say that a function $f : T \to \mathcal{X}$ is weakly Baire measurable if for every U, V open disjoint subsets of $\mathcal{X}$ the following condition holds:*

$$f^{-1}[U] \notin \mathcal{M} \wedge f^{-1}[V] \notin \mathcal{M} \Longrightarrow \left[f^{-1}[U]\right]_{\mathcal{M}} \neq \left[f^{-1}[V]\right]_{\mathcal{M}}.$$

A class of weakly Baire measurable functions is a superclass of a class of Baire measurable functions.

The next theorem is a generalization of Banach theorem about a union of open first category sets to the case of weakly Baire measurable functions.

Theorem 23 ([H1], Theorem 3.6)**.** *Assume that $\mathcal{X}$ is a metrizable space. Let $f : T \to \mathcal{X}$ be a weakly Baire measurable function. Let us consider a family*

$$\mathcal{A} = \{f^{-1}[U] : \ U \text{ is open in } \mathcal{X} \text{ and } f^{-1}[U] \in \mathcal{M}\}.$$

Then $\bigcup \mathcal{A} \in \mathcal{M}$.

An important tool in the proof of the previous theorem is Theorem 52 about completely $\mathcal{M}$-nonmeasurable union.

Theorem 24 ([H1], Theorem 3.8)**.** *Assume that $\mathcal{X}$ is a metrizable space. Let $f : T \to \mathcal{X}$ be a weakly Baire measurable function. Then there exists a set $M \in \mathcal{M}$ such that $f[T \setminus M]$ is separable.*

In the proof the important tool is Kowalsky theorem saying that each metrizable space is homeomorphic to a subspace of a hedgehog space.

Let us now consider the case of nonmetrizable space $\mathcal{X}$. We do not expect negative results in ZFC theory. Under some set-theoretic assumptions theorems about existence of a nonmeasurable union can have weak assumptions.

We will focus on a concrete model. Let us consider a generic extension of the constructible universe by adding ω_2 Solovay reals.

Lemma 4 ([H1], Claim 4.1)**.** *In $L^{Random(\omega_2)}[G]$ there exists a sequence $(K^\alpha)_{\alpha \in \omega_2}$ of meager sets such that $\bigcup_{\alpha \in A} K^\alpha = \mathbb{R}$ for every uncountable A.*

The required set K^α is defined to be the translation of meager F_σ-set of full measure by generic random real r_α.

Lemma 5 ([H1], Claim 4.2). *In $L^{Random(\omega_2)}[G]$ the following statement is true: For every σ-ideal I of subsets of ω_1 there is a point-countable family $\{K^\alpha\}_{\alpha\in\omega_1}$ such that for every $A \subseteq \omega_1$*

$$A \in I \Longrightarrow \bigcup_{\alpha\in A} K^\alpha \in \mathcal{M}$$

and

$$A \notin I \Longrightarrow \bigcup_{\alpha\in A} K^\alpha \text{ is residual.}$$

Lemma 6 ([H1], Claim 4.3 Cichoń). *In $L^{Random(\omega_2)}[G]$ the following statement is true: For every σ-ideal I of subsets of ω_1 there exists a function $g : \mathbb{R} \to \omega_1$ such that for $A \subseteq \omega_1$*

$$A \in I \Longleftrightarrow g^{-1}[A] \in \mathcal{M}.$$

Theorem 25 ([H1], Theorem 4.4). *In $L^{Random(\omega_2)}[G]$ the following statement is true: Let ω_1 be a topological space with the order topology. There exists a function $f : \mathbb{R} \to \omega_1$ satisfying the following conditions*

(1) *f is Baire measurable,*
(2) $\bigcup\{f^{-1}[U] :\ f^{-1}[U] \in \mathcal{M}$ *and U is open in the order topology*$\} = \mathbb{R}$.

The space ω_1 with the order topology is a normal space.

Images. A paper [H4] (written jointly with R. Rałowski) generalizes results from [Ky] and [CFF] concerning nonmeasurable algebraic sums in the unit interval or the Cantor cube. Generalizations consist of replacing addition with another two-valued function, or, more generally, realtion. Main theorems have quite technical formulations.

Theorem 26 ([H4], Theorem 3.1). *Let T be any set, X – an uncountable Polish space and $\mathcal{I}$ a σ-ideal with Borel base and the hole property. Let $R \subseteq X^2 \times T$ be a binary relation satysfying the following conditions:*

(1) $[R[X^2]]_{\mathcal{I}} = T$,
(2) $\{x \in T :\ \ |\{(a,b) \in X^2 :\ ((a,b),x) \in R\}| < \} \in \mathcal{I}$,
(3) $(\forall x \in T)(\forall a \in X)(|\{b \in X :\ ((a,b),x) \in R \vee ((b,a),x) \in R\}| \leq \omega)$,
(4) $(\forall x,y \in T)(x \neq y \Rightarrow |\{(a,b) \in R^{-1}[\{x\}] :$
$\{(b,a),(a,a),(a,b),(b,b)\} \cap R^{-1}[\{y\}] \neq \emptyset\}| \leq \omega)$,
(5) *there exists a cardinal $\lambda < \mathfrak{c}$ such that*
$(\forall a,b \in X)(|R(a,b)| \leq \lambda)$.

Then there exists a set $A \subseteq X$ such that $R[A \times A]$ is completely $\mathcal{I}$-nonmeasurable in T.

The following corollary of Theorem 26 generalizes a result from [CFF] and Theorem **??** from [Ky].

Corollary 1 ([H4], Corollary 3.2). *Let $f \in \mathbb{R}[x,y]$ be any symmetric polynomial of two variables with range equal to $\mathbb{R}$. Then there exists a subset $A \subset \mathbb{R}$ such that $f[A \times A]$ is Bernstein set in $\mathbb{R}$.*

Next theorem deals with a sequence of relations instead of one relation. The assumption of full hole of the image of the relation is replaced by the assumption that the complement of the image belongs to the ideal.

Theorem 27 ([H4], Theorem 3.4)**.** *Let X be any set, T – an uncountable Polish space, $\mathcal{I}$ – σ-ideal of subsets of T with Borel base. Let $(R_\alpha)_{\alpha<\mathfrak{c}} \subseteq (T^2 \times X)^{\mathfrak{c}}$ be a sequence of binary relations satisfying the following conditions:*

(1) $\{x : \ |R_\alpha^{-1}(x)| \neq \mathfrak{c}\} \in \mathcal{I}$,
(2) $|R_\alpha \cap S| < \lambda$ *for every S of the form* $\Delta \times \{x\}, \ \{a\} \times T \times \{x\}, \ T \times \{a\} \times \{x\}$, *where* $a \in X, \ x \in T$,
(3) $(\forall B \in Bor(T) \setminus \mathcal{I})(\exists a \in X) \ |R_\alpha^{-1}[B] \cap \{a\} \times T| = \mathfrak{c}$,
(4) $(\forall (a,b) \in X^2) \ |R_\alpha(a,b)| < \lambda$.

Then there exists $A \subseteq X$ such that for every $\alpha < \mathfrak{c}$ the set $R_\alpha[A \times A]$ is completely $\mathcal{I}$-nonmeasurable in T.

Theorem 27 let us obtain the following two corollaries concerning nonmeasurability with respect to the σ-ideal of null subsets and the σ-ideal of meager sets respectively.

Corollary 2 ([H4], Corollary 3.3)**.** *There exists a subset A of the real line $\mathbb{R}$ such that $f[A \times A]$ is completely $\mathcal{N}$-nonmeasurable for every surjection $f : \mathbb{R}^2 \to \mathbb{R}$ which is C^1.*

Corollary 3 ([H4], Corollary 3.4)**.** *There exists a subset A of the real line $\mathbb{R}$ such that $f[A \times A]$ is completely $\mathcal{M}$-nonmeasurable for every surjection $f : \mathbb{R}^2 \to \mathbb{R}$ which is C^1 and its partial derivatives do not vanish almost everywhere.*

Theorem 28 ([H4], Theorem 3.5)**.** *Let $X_1, \ X_2$ be any sets and T – an uncountable Polish space, $\mathcal{I}$ – σ-ideal of subsets of T with Borel base. Let $f : X_1 \times X_2 \to T$ be an any function satisfying the following conditions:*

(1) $f[X_1 \times X_2] = T$,
(2) $\{x \in T : \ \omega < |f^{-1}(x)|\} \in \mathcal{I}$,
(3) *for every Borel subset $B \in Bor(T) \setminus \mathcal{I}$ we have*

$$|\{a \in X_1 : \ \ |\{a\} \times X_2 \cap f^{-1}[B]| = \}| = ,$$

Then there exists $A \subseteq X_1$ and $B \subseteq X_2$ such that $f[A \times B]$ is completely $\mathcal{I}$-nonmeasurable in T. Moreover, if $X_1 = X_2$, then there exists $A \subset X_1$ such that $f[A \times A]$ is completely $\mathcal{I}$-nonmeasurable.

Theorem 28 together with Mycielski's theorem imply the following result.

Corollary 4 ([H4], Corollary 3.5)**.** *Let $(X,\mathcal{I}_X)$, $(Y,\mathcal{I}_Y)$ and $(Z,\mathcal{I}_Z)$ be any Polish ideal spaces. Let us assume that $f : X \times Y \mapsto Z$ is a function having the following properties:*

(1) $f[X \times Y] = Z$,
(2) $\{z \in Z : \ \ |\{(x,y) \in X \times Y : \ f(x,y) = z\}| > \omega\} \in \mathcal{I}_Z$,
(3) $(\forall B \in Bor(Z) \setminus \mathcal{I}_Z)(f^{-1}[B] \subseteq X \times Y$ *has inner positive measure with respect to the family* $Bor(X \times Y) \setminus (\mathcal{I}_X \otimes \mathcal{I}_Y))$.

Then there exist $A \subseteq X$ and $B \subseteq Y$ such that $f[A \times B]$ is completely $\mathbb{I}_Z$-nonmeasurable.

Let us recall that $\mathcal{I}_X \otimes \mathcal{I}_Y$ denotes the Fubini product of ideals $\mathcal{I}_X$ and $\mathcal{I}_Y$ i.e. σ-ideal generated by Borel subsets $A \subseteq X \times Y$ satisfying the following condition:

$$\{x \in X : \ \{y \in Y : \ (x,y) \in A\} \notin \mathcal{I}_Y\} \in \mathcal{I}_X.$$

Subsets. We will discuss results concerning some special nonmeasurable sets. Properties of these sets satisfy the notions defined below generalizing those of Luzin's and Sierpiński's set.

Definition 8. *We say that a set L is $(\mathcal{I},\mathcal{J})$-Luzin set if*

(1) $L \notin \mathcal{I}$,
(2) *for every set* $A \in \mathcal{I}$ *we have* $L \cap A \in \mathcal{J}$.

Let κ be a cardinal number. We say that L is $(\kappa,\mathcal{I},\mathcal{J})$-Luzin set, if L is $(\mathcal{I},\mathcal{J})$-Luzin set and L is of cardinality κ.

Let us notice that if $\mathcal{J}$ is equal to the family of all countable sets then we obtain the classical definition of Luzin's set (for $\mathcal{I} = \mathcal{M}$) and Sierpiński's set (for $\mathcal{I} = \mathcal{N}$). In a similar way one can obtain so called generalized Luzin sets.

Definition 9. *Let $\mathcal{F} \subseteq T^T$ be a family of functions. We say that sets A, B are equivalent with respect to $\mathcal{F}$ if:*

$$(\exists f \in \mathcal{F})(B = f[A] \vee A = f[B]).$$

Let us start with a technical result.

Theorem 29 ([H7], Theorem 2.1)**.** *Assume that $\kappa = cov(\mathcal{I}) \leq non(\mathcal{J})$. Let $\mathcal{F}$ be a family of functions mapping T into T such that $|\mathcal{F}| \leq \kappa$. Then there is a sequence $(L_\alpha)_{\alpha\in\kappa}$ such that*

(1) *L_α is a $(\kappa,\mathcal{I},\mathcal{J})$-Luzin set,*
(2) *for $\alpha \neq \beta$ sets L_α, L_β are not equivalent with respect to a family $\mathcal{F}$.*

The proof of this theorem uses transfinite induction.

Now, let us state some applications.

Corollary 2 ([H7], Corollary 2.1)**.** *Assume that $cov(\mathcal{I}) = non(\mathcal{J}) = \mathfrak{c}$. Than there are continuum many Borel not equivalent $(\mathcal{I},\mathcal{J})$-Luzin sets.*

Corollary 3 ([H7], Corollary 2.2)**.** *Assume that $cov(\mathcal{I}) = non(\mathcal{J}) = \mathfrak{c}$. Then there are continuum many $(\mathcal{I},\mathcal{J})$-Luzin sets not equivalent with respect to the family of all $\mathcal{I}$-measurable functions.*

Corollary 4 ([H7], Corollary 2.3)**.**

(1) *Assume that $cov(\mathcal{N}) = \mathfrak{c}$. Then there are continuum many $(\mathfrak{c},\mathcal{N},\mathcal{M})$-Luzin sets not equivalent with respect to the family of all Lebesgue measurable functions.*
(2) *Assume that $cov(\mathcal{M}) = \mathfrak{c}$. Then there are continuum many $(\mathfrak{c},\mathcal{M},\mathcal{N})$-Luzin sets not equivalent with respect to the family of all Baire measurable functions.*

Now, let us describe a class of forcing notions preserving $(\mathcal{I},\mathcal{J})$-Luzin sets. We are mainly interested in so called definable forcings (see [Za]), i.e. forcings of the type $Bor(T) \setminus \mathcal{I}$ for an absolutely definable σ-ideal $\mathcal{I}$. Let us start with a technical lemma.

Lemma 7 ([H7], Lemma 3.1)**.** *Assume that $\mathcal{I}$ has Fubini property. Let $\mathbb{P}_\mathcal{I} = Borel(T) \setminus \mathcal{I}$ be a definable forcing notion which is proper. Let $B \in \mathcal{I}$ be a set in $V^{\mathbb{P}_\mathcal{I}}[G]$. Then there is a set $D \in \mathcal{I} \cap V$ such that $B \cap T^V \subseteq D$.*

Theorem 30 ([H7], Theorem 3.1)**.** *Assume that κ is an uncountable cardinal number, ideals $\mathcal{I},\mathcal{J}$ have Suslin property and Fubini property. Let $\mathbb{P}_\mathcal{I} = Borel(T)\setminus\mathcal{I}$ and $\mathbb{P}_\mathcal{J} = Borel(T)\setminus\mathcal{J}$ be definable forcings. Then $\mathbb{P}_\mathcal{I}$ preserves $(\kappa,\mathcal{I},\mathcal{J})$-Luzin sets.*

Theorem 31 ([H7], Theorem 3.2). *Let $(\mathbb{P}, \leq)$ be a forcing notion such that a set*

$$\{B: \ B \in \mathcal{I} \cap Borel(T), B \text{ is coded in } V\}$$

forms a base of ideal $\mathcal{I}$ in $V^{\mathbb{P}}[G]$. Assume that Borel codes for the sets from ideals $\mathcal{I}$, $\mathcal{J}$ are absolute. Then $(\mathbb{P}, \leq)$ preserves $(\mathcal{I}, \mathcal{J})$-Luzin sets.

The latter theorem has some applications.

Corollary 5 ([H7], Corollary 3.1). *Let $(\mathbb{P}, \leq)$ be a forcing notion which does not add reals, i.e. $(\omega^\omega)^V = (\omega^\omega)^{V^{\mathbb{P}}[G]}$. Assume that Borel codes for the sets from ideals $\mathcal{I}$, $\mathcal{J}$ are absolute. Then $(\mathbb{P}, \leq)$ preserves $(\mathcal{I}, \mathcal{J})$-Luzin sets.*

Corollary 6 ([H7], Corollary 3.2). *Let $(\mathbb{P}, \leq)$ be a σ-closed forcing notion. Assume that Borel codes for sets from ideals $\mathcal{I}$, $\mathcal{J}$ are absolute. Then $(\mathbb{P}, \leq)$ preserves $(\mathcal{I}, \mathcal{J})$-Luzin sets.*

Corollary 7 ([H7], Corollary 3.3). *Let λ be an ordinal number. Let $\mathbb{P}_\lambda = ((P_\alpha, \dot{Q}_\alpha): \ \alpha < \lambda)$ be an iterated forcing with countable support. Assume that*

(1) *for every α $P_\alpha \Vdash \dot{Q}_\alpha$ is σ-closed,*
(2) *Borel codes for sets from ideals $\mathcal{I}$, $\mathcal{J}$ are absolute.*

Then $\mathbb{P}_\lambda$ preserves $(\mathcal{I}, \mathcal{J})$-Luzin sets.

In the following lemmas we will use the relation $\sqsubseteq^{random}$ and $\sqsubseteq^{Cohen}$ defined by M. Goldstern in [Go]. Let Ω be a family of clopen subsets of the Cantor space 2^ω with discrete topology. Set

$$C^{random} = \{f \in \Omega^\omega : (\forall n \in \omega)\mu(f(n)) < 2^{-n}\}.$$

For $f \in C^{random}$ set $A_f = \bigcap_{n\in\omega} \bigcup_{k\geq n} f(k)$. A relation $\sqsubseteq_n^{random}$ is defined by the formula

$$(\forall f \in C^{random})(\forall g \in 2^\omega)(f \sqsubseteq_n^{random} g \leftrightarrow (\forall k \geq n) g \notin f(k)).$$

$\sqsubseteq^{random} = \bigcup_{n\in\omega} \sqsubseteq_n^{random}$.

Let C^{Cohen} be a set of functions from $\omega^{<\omega}$ into itself. Then

$$(\forall f \in C^{Cohen})(\forall g \in \omega^\omega)(f \sqsubseteq_n^{Cohen} g \leftrightarrow (\forall k < n)(g \restriction k \frown f(g \restriction k) \subseteq g)).$$

$\sqsubseteq^{Cohen} = \bigcup_{n\in\omega} \sqsubseteq_n^{Cohen}$.

Theorem 32 ([H7], Theorem 3.4). *Assume that $\mathbb{P}$ is a forcing notion which preserves relation $\sqsubseteq^{random}$. Then $\mathbb{P}$ preserver $(\mathcal{N}, \mathcal{M})$-Luzin sets.*

Theorem 33 ([H7], Theorem 3.5). *Assume that $\mathbb{P}$ is a forcing notion which preserves relation $\sqsubseteq^{Cohen}$. Then $\mathbb{P}$ preserves $(\mathcal{M}, \mathcal{N})$-Luzin sets.*

The next corollary is based on a result of Shelah (see [Sh]) about preserving property "every new open dense set contains an old open dense set" by a countable support iteration of proper forcings with the property mentioned.

Corollary 8 ([H7], Corollary 3.4). *Let λ be an ordinal number. Let $\mathbb{P}_\lambda = ((P_\alpha, \dot{Q}_\alpha): \ \alpha < \lambda)$ be an iterated forcing with countable support. Assume that*

(1) *for every α $P_\alpha \Vdash \dot{Q}_\alpha$ is proper,*
(2) *$P_\alpha \Vdash \dot{Q}_\alpha \Vdash$ every new dense open set contains an old dense set.*

Then $\mathbb{P}_\lambda$ preserves $(\mathcal{M}, \mathcal{N})$-Luzin sets.

Results from [H8] (co-authored by M. Michalski) concern $\mathcal{I}$-Luzin sets. We will work in Euclidean space $\mathbb{R}^n$ equipped with a natural addition $+$ and multiplication $\cdot$ by real numbers. Sometimes we will treat a Euclidean space as a linear space over the field of rationals $\mathbb{Q}$. We will assume that a σ-ideal $\mathcal{I}$ is *translation invariant*, i.e. for every $\bar{x} \in \mathbb{R}^n$ and $A \in \mathcal{I}$ we have $\bar{x} + A \in \mathcal{I}$.

Definition 10. *We say that a set A is*

- *an $\mathcal{I}$-Luzin set, if for every $I \in \mathcal{I}$ we have $|A \cap I| < |A|$;*
- *a super $\mathcal{I}$-Luzin set, if A is an $\mathcal{I}$-Luzin set and for every Borel set $B \notin \mathcal{I}$ we have $|A \cap B| = |A|$.*

Let us notice that that the classical definition of generalized Luzin set coincides with the definition of $\mathcal{M}$-Luzin set, and the classical definition of generalized Sierpiński set coincides with the definition of $\mathcal{N}$-Luzin set.

Lemma 8 ([H8], Lemma 2.1)**.** *Let P, Q be perfect subsets of $\mathbb{R}^n$. Then there exist perfect subsets $P' \subseteq P$ and $Q' \subseteq Q$ such that for every $x \in \mathbb{R}^n$ we have*

$$|(P' + x) \cap Q'| \leq 1.$$

Using Lemma 8 we can construct a translation invariant σ-ideal $\mathcal{J}$ with Borel base such that there is a measurable, or even Borel, $\mathcal{J}$-Luzin set.

Theorem 34 ([H8], Theorem 2.1)**.** *There exists a translation invariant σ-ideal $\mathcal{J}$ with Borel base and a perfect set A which is a $\mathcal{J}$-Luzin set.*

Additional assumptions on ideal $\mathcal{I}$ ensure that every $\mathcal{I}$-Luzin set is $\mathcal{I}$-nonmeasurable.

The following definition was introduced in [BFN].

Definition 11. *$\mathcal{I}$ has weaker Smital property, if there exists countable dense set D such that $(A + D)^c \in \mathcal{I}$ for every Borel set $A \notin \mathcal{I}$.*

The property mentioned above is a generalization of the classical Smital property. Examples of natural σ-ideals possessing this property can be found in [BK] and [BFN].

Theorem 35 ([H8], Theorem 2.2)**.** *Assume that $\mathcal{I}$ has weaker Smital property. Then each $\mathcal{I}$-Luzin set is $\mathcal{I}$-nonmeasurable.*

In the proof of the latter theorem the crucial part is played by Lemma 8.

Lemma 9 ([H8], Lemma 2.2)**.** *Assume that $\mathcal{I}$ has weaker Smital property. Then the existence of an $\mathcal{I}$-Luzin set implies the existence of a super $\mathcal{I}$-Luzin set.*

Lemma 10 ([H8], Lemma 2.3)**.** *Let L be an $\mathcal{I}$-Luzin set. Then there exists a linearly independent $\mathcal{I}$-Luzin set.*

Lemma 11 ([H8], Lemma 2.4)**.** *Assume that $\mathcal{I}$ has weaker Smital property. Let L be an $\mathcal{I}$-Luzin set of cardinality $\mathfrak{c}$. Then there exists a linearly independent super $\mathcal{I}$-Luzin set.*

Theorem 36 ([H8], Theorem 2.3 and Theorem 2.4)**.** *Let L be a linearly independent $\mathcal{I}$-Luzin set of cardinality $\mathfrak{c}$. Then there exists a set X such that $\{x + L : x \in X\}$ is a partition of $\mathbb{R}^n$. Moreover, under CH, X can be also an $\mathcal{I}$-Luzin set.*

Theorem 37 ([H8], Theorem 2.5, 2.6, 2.7, 2.8)**.** *Assume CH.*

- *Assume that for every $A \in \mathcal{I}$ we have $\frac{1}{2}A \in \mathcal{I}$. Then there is an $\mathcal{I}$-Luzin set L such that the set $L+L$ is an $\mathcal{I}$-Luzin set.*
- *Assume that for every $A \in \mathcal{I}$ we have $-A \in \mathcal{I}$.Then there is an $\mathcal{I}$-Luzin set L such that $L+L=\mathbb{R}^n$.*
- *Assume that $\mathcal{I}$ is closed under rational scaling, i.e. $(\forall x \in \mathbb{Q})(\forall A \in \mathcal{I})(xA=\{xa : a \in A\} \in \mathcal{I})$. Then for every $m \in \omega \setminus \{0\}$ there exists an $\mathcal{I}$-Luzin set L such that $\underbrace{L+L+\cdots+L}_{m}$ is an $\mathcal{I}$-Luzin set and $\underbrace{L+L+\cdots+L}_{m+1}=\mathbb{R}^n$.*
- *Assume that $\mathcal{I}$ is closed under rational scaling. Then there exists a linearly independent $\mathcal{I}$-Luzin set L such that span(L) is an $\mathcal{I}$-Luzin set.*

Corollary 9 ([H8], Corollary 2.1)**.** *Assume CH. Let $\mathcal{I}$ be closed under scaling. Then*

(1) *there exists an $\mathcal{I}$-Luzin set L such that $\underbrace{L+L+\cdots+L}_{n+1}$ is an $\mathcal{I}$-Luzin set for every $n \in \omega$,*
(2) *there exists an $\mathcal{I}$-Luzin set L such that $L+L=L$,*
(3) *there exists an $\mathcal{I}$-Luzin set L such that $\langle\underbrace{L+L+\cdots+L}_{n+1} : n \in \omega\rangle$ is ascending sequence of $\mathcal{I}$-Luzin sets.*

Theorem 38 ([H8], Theorem 2.9)**.** *It is consistent with ZFC that $\mathfrak{c}=\omega_2$ and there exists a Luzin set which is a linear subspace of $\mathbb{R}^n$.*

In the proof of this theorem we consider a model which is a generic extension of a model of CH by adding ω_2 Cohen reals. The set from the conclusion of the theorem is a subspace generated by added generic Cohen reals.

Theorem 39 ([H8], Theorem 2.10)**.** *Assume CH. There exists a Luzin set L such that $L+L$ is a Bernstein set.*

The proof uses a standard transfinite induction. To show that finding a suitable object at a step α is possible we find a Σ^1_2 sentence which implies it. We show that this sentence is true in a generic extension of our model and we use Shoenfield theorem to prove that it is true in the model we have started with.

Theorem 40 ([H8], Theorem 2.11)**.** *Assume CH. There exists a Sierpiński set S such that $S+S$ is a Bernstein set.*

Next few lemmas show a connection of the algebraic structure of the space with the topological structure and the measure structure.

Lemma 12 ([H8], Lemma 2.5)**.** *There exists a residual null set R and a nowhere dense perfect null set P such that $R+P \subseteq R$.*

This result can be seen as a generalization of a result from [Re], where I. Recław showed that for every null set A and every perfect set P there exists a perfect subset $P' \subseteq P$ such that $A+P$ has measure zero. In fact, the following stronger result is also true.

Lemma 13 ([H8], Lemma 2.6)**.** *Let A be a null set. There exists a perfect set P such that for every n*

$$A+\underbrace{P+P+\cdots+P}_{n} \in \mathcal{N}.$$

Lemma 14 ([H8], Lemma 2.8)**.** *For every compact null set P there exist a residual G_δ G such that a set $G+P$ is null.*

In [BS] L. Babinkostova, M. Scheepers showed that for any Luzin set L and any Sierpiński set S, the set $L \times S$ is Menger, and so is $L+S$. Menger sets are not Bernstein, thus $L+S$ is not a Bernstein set. Moreover in [Sc] M. Scheepers proved that if A is null and has property s_0 and S is a Sierpiński set then $A+S$ has property s_0. Next theorem slightly generalizes these results under the hypothesis that $\mathfrak{c}$ is a regular cardinal.

Theorem 41 ([H8], Theorem 2.12)**.** *Assume that $\mathfrak{c}$ is a regular cardinal. For every generalized Luzin set L and for every generalized Sierpiński set S, a set $L+S$ has property s_0.*

The proof of this theorem uses Lemma 14.

The next theorem shows that the assumption of regularity of $\mathfrak{c}$ cannot be omitted.

Theorem 42 ([H8], Theorem 2.13)**.** *The following sentence is consistent with ZFC: There exist a generalized Luzin set L and a generalized Sierpiński set S such that $L+S=\mathbb{R}^n$.*

The construction of the required model is done in the following steps. We start with a model of GCH. Let P_α be a finite support iteration of $(\dot{Q}_\beta :\ \beta < \alpha)$, where $\Vdash_\beta \dot{Q}_\beta = \mathcal{C} \times \mathcal{R}_{\aleph_{\beta+2}}$ for $\beta < \alpha$. $\mathcal{C}$ denotes the forcing notion adding one Cohen real (as an element of $\mathbb{R}^n$) and $\mathcal{R}_\kappa$ is the forcing notion adding κ independent Solovay reals (as elements of $\mathbb{R}^n$). In the final model we define the required sets in the following way

$$S = \bigcup_{\alpha\in\omega_1} R_\alpha \cup \{-c_\alpha\}, \quad L = \bigcup_{\alpha\in\omega_1} \{c_\alpha + x :\ x \in \mathbb{R}^n \cap V_\alpha\}.$$

Other research achievements

A list of papers in other research achievements:

[P1] Sz. Żeberski, *Nonstandard proofs of Eggelston like theorems*, Proceedings of the Ninth Prague Topological Symposium (Prague 2001) Topology Atlas (Toronto 2002), 353-357,

[P2] J.Cichoń, Sz. Żeberski, *On the splitting number and Mazurkiewicz theorem*, Acta Universitatis Carolinae, Mathematica et Physica 42 (2) (2001), 23-25,

[P3] J.Cichoń, M.Morayne, R.Rałowski, C.Ryll-Nardzewski, Sz. Żeberski, *On nonmeasurable unions*, Topology and its Applications 154 (2007), 884-893,

[P4] R. Rałowski, P. Szczepaniak, Sz. Żeberski, *A generalization of Steinhaus theorem and some nonmeasurable sets*, Real Analysis Exchange, 35 (1) (2009/2010), 1-9,

[P5] J. Kraszewski, R. Rałowski, P. Szczepaniak, Sz. Żeberski, *Bernstein sets and kappa coverings*, Mathematical Logic Quarterly, 56 (2) (2010), 216-224,

[P6] M. Bienias, Sz. Głąb, R. Rałowski, Sz. Żeberski, *Two point sets with additional properties*, Czechoslovak Mathematical Journal, 63 (4) (2013), 1019-1037,

[P7] T. Banakh, M. Morayne, R. Rałowski, Sz. Żeberski, *Topologically invariant σ-ideals on Euclidean spaces*, Fundamenta Mathematicae, 231 (2015), 101-112,

[P8] T. Banakh, M. Morayne, R. Rałowski, Sz. Żeberski, *Topologically invariant σ-ideals on the Hilbert cube*, Israel Journal of Mathematics, 209 (2015), 715–743,

[P9] T. Banakh, R. Rałowski, Sz. Żeberski, *Classifying invariant σ-ideals with analytic base on good Cantor measure spaces*, accepted to Proceedings of the American Mathematical Society.

The paper [P1] contains a nonstandard proof of the Eggelston theorem which says that every subset of the plane of positive Lebesgue measure contains a product of two perfect sets. The proof is based on Shoenfield theorem about Σ^1_2-absoluteness and theorem from [CKP] saying that a Boolean algebra $Borel/\mathcal{N}$ contains a dense subset of cardinality $cof(\mathcal{N})$.

Moreover, using similar techniques, the following result was proved.

Theorem 43 ([P1], Theorem 4)**.** *Let $A \subseteq [0,1]^2$ be of measure 1. Then we can find two sets $F, Q \subseteq [0,1]$ satisfying conditions: F is F_σ-set of measure 1, Q is a perfect set and $F \times Q \subseteq A$.*

The analogous results concerning the ideal of meager sets were also showed.

Theorem 44 ([P1], Theorem 5)**.** *Let $A \subseteq [0,1]^2$ be a nonmeager that has Baire property. Then we can find two sets $G, Q \subseteq [0,1]$ satisfying conditions: G is G_δ-set, $G \notin \mathcal{M}$, Q is a perfect set and $G \times Q \subseteq A$.*

Theorem 45 ([P1], Theorem 6)**.** *Let $A \subseteq [0,1]^2$ be a residual set. Then there exist two sets $C, Q \subseteq [0,1]$ satisfying conditions: C is a dense G_δ, Q is a perfect set and $C \times Q \subseteq A$.*

In [P2] (written jointly with J. Cichoń) a nonstandard proof of Mazurkiewicz theorem was given. Mazurkiewicz theorem says that for every sequence $\{f_n\}_{n\in\omega}$ of Borel functions from $[0,1]$ into $[0,1]$ we can choose a subsequence point-wise converging on some perfect subset. The proof is based on Shoenfield absoluteness theorem and the following characterization of a splitting number $\mathfrak{s}$.

Lemma 15 ([P2], Lemma 1). *The following cardinal numbers are all equal:*

(1) $\mathfrak{s} = \min\{\kappa :\ \{0,1\}^\kappa$ *is not sequentially compact* $\}$
(2) $\mathfrak{s}' = \min\{\kappa :\ (\{0,1\}^\omega)^\kappa$ *is not sequentially compact* $\}$
(3) $\mathfrak{s}'' = \min\{\kappa :\ [0,1]^\kappa$ *is not sequentially compact* $\}$

In [P3] (co-authored by J. Cichoń, M. Morayne, R. Rałowski and Cz. Ryll-Nardzewski) results concerning finding subfamilies with completely $\mathcal{I}$-nonmeasurable union were obtained. Additional assumptions involving cardinal coefficients of $\mathcal{I}$ were used. These assumptions for many natural σ-ideals are independent of ZFC.

Theorem 46 ([P3], Theorem 3.2). *Assume that $\mathcal{I}$ is a σ-ideal with Borel base on a Polish space T and* $\mathrm{cov}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$. *Let $\mathcal{A} \subseteq \mathcal{I}$ be a family such that $T \setminus \bigcup \mathcal{A} \in \mathcal{I}$. Let $\mathcal{A}$ be point small, i.e.*

$$\left\{x :\in T :\ \bigcup\{A \in \mathcal{A} :\ x \notin A\} \notin \mathcal{I}\right\} \in \mathcal{I}.$$

Then there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ with completely $\mathcal{I}$-nonmeasurable union $\bigcup \mathcal{A}'$.

Theorem 47 ([P3], Theorem 3.1). *Let $\mathcal{I}$ be a σ-ideal on a Polish space T such that there exists a completely $\mathcal{I}$-nonmeasurable set of cardinality smaller than $cov_h(\mathcal{I})$. Then for every family $\mathcal{A} \subseteq \mathcal{I}$ such that $T \setminus \bigcup \mathcal{A} \in \mathcal{I}$ there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ with completely $\mathcal{I}$-nonmeasurable union $\bigcup \mathcal{A}'$.*

For the σ-ideal of meager subsets of the real line $\mathbb{R}$ the hypothesis of Theorem 47 is fulfilled in the generic extension of the constructible universe L by ω_2 independent Cohen reals. Then $cov(\mathcal{M}) = \omega_2 = \mathfrak{c}$ and the required completely nonmeasurable set is of the form:

$$\{c_\xi + r \in \mathbb{R} :\ \xi < \omega_1 \wedge r \in \mathbb{Q}\}.$$

Similar argument works in the case of the null ideal $\mathcal{N}$, if we add ω_2 independent random reals to L.

Theorem 47 has an application in the paper [Kuz] of Y. Kuznetzova connected with harmonic analysis. Kuznetzova asked the question if for every nonempty null set $A \in \mathcal{N}$ there exists a set B such that the algebraic sum $A + B$ is nonmeasurable. In every model, where the conclusion of Theorem 47 is true for measure (e.g. in the model obtained by adding ω_2 random reals to L) the answer to Kuznetzova's question is positive. As a consequence, every measurable homomorphism between a locally compact group and a topological group is continuous.

Theorem 46 was used to find nonmeasurable unions in some families of subsets of abelian Polish groups with translation invariant σ-ideals. We say that a set $C \subseteq G$ is $\mathcal{I}$-Gruenhage if for every $\mathcal{I}$-measurable set B and for every set $T \in [G]^{<\mathfrak{c}}$, the set $B \setminus (C + T)$ is nonempty. Darji and Keleti [DK] proved that if $C \subseteq \mathbb{R}$ is a compact set with packing dimension $dim_p(C) < 1$ then $\mathbb{R} \neq T + C$ for $T \in [\mathbb{R}]^{<\mathfrak{c}}$. The classical Cantor set C is an example of $\mathcal{N}$-Gruenhage set.

Theorem 48 ([P3], Theorem 5.2). *If $\mathcal{I}$ is a translation invariant σ-ideal with Borel base on an abelian Polish group $(G, +)$ then for every set $C \subseteq G$, such that $C \cup -C$ is an $\mathcal{I}$-Gruenhage set there exists $P \subseteq G$ such that $P + C$ is a completely $\mathcal{I}$-nonmeasurable set in G.*

It is natural to ask if the conclusion of this theorem can be strengthened to $P \subseteq C$. The answer is positive for the classical Cantor set. The following theorem was proved using the ultrafilter method.

Theorem 49 ([P3], Corollary 5.10). *If C is the classical Cantor set then there exists its subset $P \subseteq C$ such that the algebraic sum $P + C$ is Lebesgue nonmeasurable.*

Theorem 50 ([P3], Theorem 4.1). *Assume that T is an uncountable Polish space and a family $\mathcal{A} \subseteq [T]^{\leq\omega}$ is point-countable, i.e. for every $x \in T$ $\{A \in \mathcal{A} : x \in A\} \in [\mathcal{A}]^{\leq\omega}$. If $\bigcup \mathcal{A} = T$ then there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is a Bernstein set.*

Theorem 51 ([P3], Theorem 4.4). *Let $\mathcal{I}$ be a σ-ideal with Borel base on a Polish space T. If $\mathcal{A} \subseteq P(T)$ is $Bor(T)[\mathcal{I}]$-sumable family of countable closed sets of countable bounded Cantor-Bendixon rank, i.e.*

$$(\exists \alpha < \omega_1)(\forall A \in \mathcal{A})\ (A^\alpha = \emptyset),$$

then $\bigcup \mathcal{A} \in \mathcal{I}$.

As a corollary we obtain that if $\mathcal{A}$ is a family of countable closed sets of countable bounded Cantor-Bendixon rank such that $\bigcup \mathcal{A} \notin \mathcal{I}$ then there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ with $\mathcal{I}$-nonmeasurable union $\bigcup \mathcal{A}'$.

Theorem 52 ([P3], Theorem 6.8). *Every partition of the real line into meager sets contains a subfamily with completely $\mathcal{M}$-nonmeasurable union.*

The proof of Theorem 52 uses Gitik Shelah theorem from [GS1] saying that Boolean algebra of the type $P(\kappa)/\mathcal{I}$ cannot be isomorphic to the Cohen algebra and Erdős-Alaoglu theorem or, more precisely, its version from [Ta].

In [P5] (written jointly with J. Kraszewski, R. Rałowski and P. Szczepaniak) authors consider subsets of abelian Polish groups which cover some translation of every set of cardinality κ (for a fixed cardinal κ). Such sets are called κ-covering. Analogously, we say that a set A contained in a group G is $< \kappa$-covering if every subset of G of cardinality less than κ can be translated into A. An inspiration for this research were results obtained by K. Muthuvel in [Mu] concerning mainly κ-covering sets for a finite cardinal number κ and results of A. Nowik from [No1, No2], where the author considers ω- or $< \omega$-covering sets of low descriptive class.

Theorem 53 ([P5], Theorem 2.1). *There exists a partition of the real line $\mathbb{R}$ into two Bernstein sets such that none of them is a 2-covering set.*

Theorem 54 ([P5], Theorem 2.2, Proposition 2.5). *There exists a partition of the real line $\mathbb{R}$ into continuum many Bernstein sets such that each of them is $< cof(\mathfrak{c})$-covering set.*

Moreover if $\mathcal{I}$ is a σ-ideal on the real line $\mathbb{R}$ having Steinhaus property then the assumption $\mathrm{non}(\mathcal{I}) < \mathfrak{c}$ *implies the existence of completely $\mathcal{I}$-nonmeasurable set which is $< \mathfrak{c}$-covering.*

The notion of κ-covering set has natural generalizations, namely S-covering and I-covering.

Definition 12. *We say that a family $\mathcal{A}$ is κ-S-covering if*

- *is a family of pairwise disjoint subsets of the real line $\mathbb{R}$,*
- $|\mathcal{A}| = \kappa$,
- $(\forall F \in [\mathbb{R}]^\kappa)(\exists t \in \mathbb{R})(\forall A \in \mathcal{A})\ |(t + F) \cap A| = 1$.

The results of [P5] concern families $\mathcal{A}$ whose elements are completely nonmeasurable sets with respect to some σ-ideals on $\mathbb{R}$. Let us state the following example.

Theorem 55 ([P5], Theorem 3.3). *Let κ be a cardinal number, $2 < \kappa < \mathfrak{c}$. If $2^\kappa \leq \mathfrak{c}$ then there exists a partition of $\mathbb{R}$ into Bernstein sets $\{B_\xi : \xi < \kappa\}$ such that*

- *for every $\xi < \kappa$ B_ξ is not a 2-covering set, but*
- $\{B_\xi :\ \xi < \kappa\}$ *is a κ-S-covering.*

MA implies that if $\omega \leq \kappa < \mathfrak{c}$ then $2^\kappa = \mathfrak{c}$, which guaranties that the conclusion of Theorem 55 is consistent with ZFC.

A slightly more general context can be found in the following result.

Theorem 56 ([P5], Theorem 3.5). *Let κ be a cardinal number such that $2^\kappa = \mathfrak{c}$. Let $(G, +)$ be an uncountable abelian Polish group with a metric d. Moreover, let $\mathcal{I} \subseteq P(G)$ be a σ-ideal on G satisfying the following conditions*

- $(\forall B \in Bor(G) \setminus \mathcal{I})(\forall \mathcal{D} \in [\mathcal{I}]^{<\mathfrak{c}})\ |B \setminus \bigcup \mathcal{D}| = \mathfrak{c}$,
- *there exists $a \in rng(d) \setminus \{0\}$ such that $(\forall x \in G)\ \{y \in G :\ d(x, y) = a\} \in \mathcal{I}$.*

Then there exists a family $\{B_\xi : \xi < \kappa\}$ of pairwise disjoint sets such that

1. *B_ξ is a completely $\mathcal{I}$-nonmeasurable subset of G for every $\xi < \kappa$,*
2. *B_ξ is not a 2-covering set for any $\xi < \kappa$,*
3. *$\{B_\xi : \xi < \kappa\}$ is κ-S-covering.*

The translation in the definition of κ-covering set can be replaced by any isometry e.g. of the plain. Let us introduce a notion called κ-I-covering set.

Definition 13. *A subset of the plain $A \subseteq \mathbb{R}^2$ is a κ-I-covering set if*

$$(\forall B \in [\mathbb{R}^2]^\kappa)(\forall \varphi)(\varphi : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is an isometry and } \varphi[B] \subseteq A).$$

The next two theorems show the difference between 2-I-covering and 3-I-covering sets.

Theorem 57 ([P5], Theorem 4.3). *Every Bernstein set in $\mathbb{R}^2$ is a 2-I-covering set.*

Theorem 58 ([P5], Theorem 4.4). *There exists a Bernstein set in $\mathbb{R}^2$ which is not a 3-I-covering set.*

The conclusion of Theorem 57 cannot be generalized to any completely $\mathcal{I}$-nonmeasurable sets.

Theorem 59 ([P5], Theorem 4.5). *If $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}\}$ then there exists a completely $\mathcal{I}$-nonmeasurable set in $\mathbb{R}^2$ which is not a 2-I-covering set.*

The aim of a paper [P6] (co-authored by M. Bienias, Sz. Głąb and R. Rałowski) was examining sets defined by S. Mazurkiewicz. These sets are known in the literature as Mazurkiewicz sets or two-point sets.

Definition 14. *We say that a subset of the plane $A \subseteq \mathbb{R}^2$ is a Mazurkiewicz set if its intersection with every line has cardinality 2.*

It is known that Mazurkiewicz sets are quite complicated. D. Larman in [La] showed that a Mazurkiewicz set cannot be of type F_σ. A. Miller in [Mi] constructed co-analytic Mazurkiewicz set in L.

Theorem 60 ([P6], Theorem 2.2). *Let $\mathcal{I}$ be a σ-ideal with Borel base containing singletons. There exists a Mazurkiewicz set M which is a Hamel base of the space $\mathbb{R}^2$ over the field of rationals $\mathbb{Q}$ and M is a completely $\mathcal{I}$-nonmeasurable set.*

Theorem 61 ([P6], Theorem 3.3). *There is a Mazurkiewicz set which belongs to the Marczewski ideal s_0.*

Theorem 62 ([P6], Theorem 3.5). *Assume that for every Borel set B outside σ-ideal $\mathcal{I}$ there exist $\mathfrak{c}$ many parallel lines and each of them intersects B in continuum many points. Then*

(1) *there exists a Mazurkiewicz set A belonging to s_0 which is a Hamel base and A is a completely $\mathcal{I}$-nonmeasurable set,*
(2) *there exists a Mazurkiewicz set B which is a Hamel base and is completely $\mathcal{I}$-nonmeasurable and is not measurable in the sense of Marczewski.*

The notion of Mazurkiewicz set can be naturally generalized to κ-point set for given cardinal number $2 \leq \kappa \leq \mathfrak{c}$. A κ-point set is a set which intersects every line in κ many points.

Theorem 63 ([P6], Theorem 4.6). *Let $n \geq 2$ be a natural number. Every n-point set can be partitioned into n pairwise disjoint bijections from $\mathbb{R}$ to $\mathbb{R}$.*

The following theorem shows a connection between Bernstein and Mazurkiewicz sets.

Theorem 64 ([P6], Theorem 4.9). *For every Bernstein set $B \subseteq \mathbb{R}$ there exists a Mazurkiewicz set $A \subseteq \mathbb{R}^2$ which is of measure zero and of first category in $\mathbb{R}^2$ such that for every function $f \subseteq A$, preimage $f^{-1}[(0,1)]$ is equal to B.*

Mazurkiewicz set can be neither Bernstein set nor Luzin set, nor Sierpiński set. This fact was a motivation to define partial Mazurkiewicz set, i.e. a set which has at most two point intersection with every line.

Theorem 65 ([P6], Theorem 5.5). *Assume CH. Then there exists a Luzin set which is a partial Mazurkiewicz set.*

The analogous result is true for Sierpiński's set.

In the next theorems connection between κ-point sets and λ-covering sets was examined.

Theorem 66 ([P6], Theorem 6.3, 6.4).
(1) *There exists an ω-point set which is not a 2-I-covering set.*
(2) *There exists an ω-point set which is an ω-covering set.*

Theorem 67 ([P6], Theorem 6.5). *Consider a model of ZFC obtained by adding ω_2 independent Cohen reals to the constructible universe L. In this model $\omega_1 < \mathfrak{c} = \omega_2$ and there exists ω_1-point set which is an ω_1-covering set.*

Theorem 68 ([P6], Theorem 6.7). *Let $n > 1$ be a natural number. Then*

(1) *there exists an n-point set A which is not a 2-I-covering set,*
(2) *there exists an n-point set B which is an n-covering set.*

In [P6], combinatorial properties of Mazurkiewicz sets from the point of view of families of almost disjoint sets were also considered.

Theorem 69 ([P6], Theorem 7.1). *Let h be a fixed definable Borel bijection between the real line $\mathbb{R}$ and the Ramsey space $[\omega]^\omega$. Let π_1, π_2 be the orthogonal projections of the plain $\mathbb{R}^2$ on the first coordinate, on the second coordinate. It is relatively consistent with ZFC that $\neg CH$ and there exists a partial Mazurkiewicz set $A \subseteq \mathbb{R}^2$ such that $h[\pi_1[A] \cup \pi_2[A]]$ is a maximal almost disjoint family of cardinality ω_1.*

Theorem 70 ([P6], Theorem 7.5). *In the model obtained by adding ω_2 Cohen reals to the constructible universe L there exists a partial Mazurkiewicz set $C \subseteq \mathbb{R}^2$ of cardinality ω_2 which is a Luzin set and fulfills the following condition*

$$(\exists A \in \mathcal{N})(\forall D \in [C]^{\omega_1})\ A + D = \mathbb{R}^2.$$

The analogous result concerning Sierpiński set can be proved in a model obtained by adding ω_2 independent Solovay reals.

An inspiration for the paper [P4] (written jointly with R. Rałowski and P. Szczepaniak) was classical Steinhaus theorem saying that for every $A, B \subseteq \mathbb{R}$ of positive Lebesgue measure, an algebraic sum $A + B$ has nonempty interior.

The following result generalizes the Steinhaus theorem mentioned above.

Theorem 71 ([P4], Theorem 2.1). *Let $\mathcal{I}$ be equal to $\mathcal{N}$ or $\mathcal{M}$. Assume that a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 and*

$$\left\{(x, y) \in \mathbb{R}^2 :\ \frac{\partial f}{\partial x}(x, y) = 0 \vee \frac{\partial f}{\partial y}(x, y) = 0\right\} \in \mathcal{I}.$$

Let $A, B \in Bor(\mathbb{R}) \setminus \mathcal{I}$ be Borel sets outside σ-ideal $\mathcal{I}$. Then the set $f[A \times B]$ contains nonempty open interval on the real line $\mathbb{R}$.

Steinhaus theorem, in its original version, was an important tool in a proof of Cichoń-Szczepaniak theorem (see [CS]) about a ball in euclidean space.

Theorem 72 (Cichoń-Szczepaniak). *Let m, n be two different positive natural numbers and $f : \mathbb{R}^n \to \mathbb{R}^m$ be an isomorphism between linear spaces over the field of rationals $\mathbb{Q}$. If a set $A \subseteq \mathbb{R}^n$ and its complement have nonempty interior in $\mathbb{R}^n$ then the image $f[A] \subseteq \mathbb{R}^m$ has inner Lebesgue measure zero and a full outer Lebesgue measure in $\mathbb{R}^m$.*

Theorem 72 was a tool to obtain nonmeasurable sets with some algebraic properties. Examples of such applications are the following theorems from discussed paper.

Theorem 73 ([P4], Theorem 3.2). *There exists completely $\mathcal{N}$-nonmeasurable set $A \subseteq \mathbb{R}$ such that $A + A = A$ and $A - A = \mathbb{R}$.*

Theorem 74 ([P4], Theorem 3.3). *There exists a partition $\mathcal{A} = \{A_n : n \in \omega\}$ of the real line $\mathbb{R}$ into completely $\mathcal{N}$-nonmeasurable sets such that for every $n \in \omega$ we have $A_n + A_n = A_n$.*

Theorem 75 ([P4], Theorem 3.5). *There exists a partition $\mathcal{A} = \{A_n : n \in \omega\}$ of the real line $\mathbb{R}$ into completely $\mathcal{N}$-nonmeasurable sets such that*

$$(\forall m, n \in \omega)\ m \neq n \Rightarrow A_m + A_n = \mathbb{R} \setminus \{0\}.$$

There are versions of these theorems for finite partitions of the real line $\mathbb{R}$.

Theorem 76 ([P4], Theorem 3.7). *There exists a set $A \subseteq \mathbb{R}$ such that $A, A+A, A+A+A, \ldots$ are completely $\mathcal{N}$-nonmeasurable and $\bigcup_{n\in\omega} \underbrace{A + \ldots + A}_{n} = \mathbb{R}$.*

The latter theorem has also its finite version.

Theorem 77 ([P4], Theorem 3.9). *There exists a set $A \subseteq \mathbb{R}$ such that*

$$A \subsetneq A + A \subsetneq A + A + A \subsetneq \ldots$$

are completely $\mathcal{N}$-nonmeasurable and $\bigcup_{n\in\omega} \underbrace{A + \ldots + A}_{n}$ is completely $\mathcal{N}$-nonmeasurable in the real line $\mathbb{R}$.

Theorem 78 ([P4], Theorem 3.11)**.** *There exist a set $A \subseteq \mathbb{R}$ such that*

$$A \supsetneq A + A \supsetneq A + A + A \supsetneq \dots$$

are completely $\mathcal{N}$-nonmeasurable in the real line $\mathbb{R}$.

In [P4] there are also multiplicative versions of previously cited results.

Theorem 79 ([P4], Corollary 3.2)**.** *There exist a completely $\mathcal{N}$-nonmeasurable set $A \subseteq \mathbb{R}$ such that $A \cdot A = A$.*

Theorem 80 ([P4], Corollary 3.3)**.** *There exists a set $A \subseteq \mathbb{R}$ such that*

$$A \subsetneq A \cdot A \subsetneq A \cdot A \cdot A \subsetneq \dots$$

are completely $\mathcal{N}$-nonmeasurable and $\bigcup_{n \in \omega} \underbrace{A \cdot \ldots \cdot A}_{n}$ is completely $\mathcal{N}$-nonmeasurable set in the real line $\mathbb{R}$.

Ideals. In [P7] (co-authored by T. Banakh, M. Morayne and R. Rałowski), nontrivial topologically invariant σ-ideals in euclidean space $\mathbb{R}^n$ were considered. We say that ideal is topologically invariant if for every set $A \in \mathcal{I}$ and for every homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ the image $h[A]$ belongs to ideal $\mathcal{I}$. σ-ideal $\mathcal{I}$ is called nontrivial if $\mathbb{R}^n \notin \mathcal{I}$ and there exists an uncountable set in $\mathcal{I}$.

Theorem 81 ([P7], Theorem 2.1)**.** *Every nontrivial σ-ideal $\mathcal{I}$ with BP-base of Euclidean space $\mathbb{R}^n$ is contained in ideal $\mathcal{M}$ of meager sets.*

From the latter theorem follows that σ-ideal $\mathcal{M}$ is the largest σ-ideal considered.

The smallest σ-ideal in topologically invariant nontrivial σ-ideals with Borel base were also described. Namely, it is σ-ideal $\sigma\mathcal{C}_0$ generated by tame Cantor sets. A Cantor set C is called *tame Cantor set* if there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h[C]$ is contained in the line $\mathbb{R} \times \{\bar{0}\}$.

Theorem 82 ([P7], Theorem 2.2)**.** *The σ-ideal $\sigma\mathcal{C}_0$ is contained in every nontrivial topologically invariant σ-ideal $\mathcal{I}$ with an analytic base in $\mathbb{R}^n$.*

Let us recall that for any ideals $\mathcal{I}$, $\mathcal{J}$ on a Polish space, relative coefficients are defined in the following way

$$\mathrm{add}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{J}\},$$

$$\mathrm{cof}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{J} \wedge (\forall A \in \mathcal{I})(\exists B \in \mathcal{B}) A \subseteq B\}.$$

The next theorem describes cardinal coefficients of the σ-ideal $\sigma\mathcal{C}_0$.

Theorem 83 ([P7], Theorem 2.4)**.** *The following equalities are true*

1. $\mathrm{cov}(\sigma\mathcal{C}_0) = \mathrm{cov}(\mathcal{M})$,
2. $\mathrm{non}(\sigma\mathcal{C}_0) = \mathrm{non}(\mathcal{M})$,
3. $\mathrm{add}(\sigma\mathcal{C}_0) = \mathrm{add}(\sigma\mathcal{C}_0, \mathcal{M}) = \mathrm{add}(\mathcal{M})$,
4. $\mathrm{cof}(\sigma\mathcal{C}_0) = \mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M}) = \mathrm{cof}(\mathcal{M})$.

The latter result allows us to calculate or at least estimate cardinal coefficients of any topologically invariant nontrivial σ-ideal $\mathcal{I}$.

Corollary 10 ([P7], Corollary 2.5)**.** *Let $\mathcal{I}$ be topologically invariant nontrivial σ-ideal of subsets of $\mathbb{R}^n$ with analytic base. Then*

(1) $\mathrm{cov}(\mathcal{I}) = \mathrm{cov}(\mathcal{M})$,
(2) $\mathrm{non}(\mathcal{I}) = \mathrm{non}(\mathcal{M})$,
(3) $\mathrm{add}(\mathcal{I}) \le \mathrm{add}(\mathcal{M})$,
(4) $\mathrm{cof}(\mathcal{I}) \ge \mathrm{cof}(\mathcal{M})$.

Thus, for $X = \mathbb{R}^n$, the following variant of Cichoń's diagram describes relations between cardinal characteristics of the ideal $\mathcal{M}$ and any nontrivial topologically invariant σ-ideal $\mathcal{I}$ ($a \to b$ stands for $a \le b$):

$$\begin{tikzcd}
& \mathrm{non}(\mathcal{I}) \arrow[r, equal] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{I}) \arrow[r] & \mathfrak{c} \\
\omega_1 \arrow[r] & \mathrm{add}(\mathcal{I}) \arrow[r] \arrow[u] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r, equal] \arrow[u] & \mathrm{cov}(\mathcal{I}) \arrow[u] &
\end{tikzcd}$$

The following example shows that the inequalities $\mathrm{add}(\mathcal{I}) \le \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M}) \le \mathrm{cof}(\mathcal{I})$ can be strict.

Example 1 ([P7], Example 2.6). *The σ-ideal $\mathcal{I}_{\mathbb{I}} \subseteq P(\mathbb{R}^2)$ generated by the interval $\mathbb{I} = [0,1] \times \{0\}$ in the plane $\mathbb{R}^2$ has cardinal characteristics:*

$$\mathrm{add}(\mathcal{I}_{\mathbb{I}}) = \omega_1, \;\; \mathrm{non}(\mathcal{I}_{\mathbb{I}}) = \mathrm{non}(\mathcal{M}), \;\; \mathrm{cov}(\mathcal{I}_{\mathbb{I}}) = \mathrm{cov}(\mathcal{M}), \;\; \textit{and} \;\; \mathrm{cof}(\mathcal{I}_{\mathbb{I}}) = \mathfrak{c}.$$

It turns out that cardinal characteristics for some class of ideals coincide with the respective cardinal characteristics of the ideal $\mathcal{M}$ of meager subsets of $\mathbb{R}^n$.

Theorem 84 ([P7], Theorem 2.7). *For every number $0 \le k < n$ the σ-ideal $\sigma\mathcal{D}_k$ generated by closed at most k-dimensional subsets of $\mathbb{R}^n$ has cardinal characteristics:*

$$\mathrm{add}(\sigma\mathcal{D}_k) = \mathrm{add}(\mathcal{M}), \;\; \mathrm{cov}(\sigma\mathcal{D}_k) = \mathrm{cov}(\mathcal{M}),$$
$$\mathrm{non}(\sigma\mathcal{D}_k) = \mathrm{non}(\mathcal{M}), \;\; \mathrm{cof}(\sigma\mathcal{D}_k) = \mathrm{cof}(\mathcal{M}).$$

A motivation in [P8] (written jointly with T. Banakh, M. Morayne and R. Rałowski) was a question posed on the web page of M. Csärnyei, if the minimal cardinality of the family of Cantor sets covering the Hilbert cube $\mathbb{I}^\omega$ is the same as the minimal cardinality of the family of Cantor sets covering the unit interval $\mathbb{I} = [0,1]$. Paper [P8] gives a positive answer.

In this paper topologically invariant nontrivial σ-ideals on the Hilbert cube $\mathbb{I}^\omega$ equipped with product topology were considered. Next theorem describes the role of the ideal $\mathcal{M}$ of meager subsets of $\mathbb{I}^\omega$.

Theorem 85 ([P8], Theorem 1.1). *The ideal $\mathcal{M}$ of meager subsets of the Hilbert cube $\mathbb{I}^\omega$ is:*

(1) *a maximal nontrivial topologically invariant ideal with BP-base on $\mathbb{I}^\omega$, and*
(2) *the largest nontrivial topologically invariant ideal with σ-compact base on $\mathbb{I}^\omega$.*

In [P8] it was proved that the family of all nontrivial topologically invariant σ-ideals with analytic base on the Hilbert cube $\mathbb{I}^\omega$ contains the smallest element, namely the σ-ideal $\sigma\mathcal{C}_0$ generated by so called tame Cantor sets in $\mathbb{I}^\omega$. So, the smallest element is described in the same way as in the case of Euclidean space $\mathbb{R}^n$. Observe that in the Hilbert cube, $\sigma\mathcal{C}_0$ coincides with the σ-ideal generated by zero-dimensional Z-sets in $\mathbb{I}^\omega$. A closed subset A of a topological space X is called a *Z-set* in X if for any open cover $\mathcal{U}$ of X there is a continuous map $f : X \to X \setminus A$, which is *$\mathcal{U}$-near* to the identity map in the sense that for each $x \in X$ the set $\{f(x), x\}$ is contained in some set $U \in \mathcal{U}$.

Theorem 86 ([P8], Theorem 1.2)**.** *The σ-ideal $\sigma\mathcal{C}_0$ is contained in each topologically invariant nontrivial (σ-)ideal $\mathcal{I}$ with analytic base on $\mathbb{I}^\omega$.*

In the study of cardinal coefficients of σ-ideal $\sigma\mathcal{C}_0$ an important tools were combinatorial descriptions of $\mathrm{cov}(\mathcal{M}), \mathrm{non}(\mathcal{M})$ given by T. Bartoszyński (see e.g. [Ba],[BJ]). Concerning add and cof the main tool was the fact that the space of all homeomorphisms on the Hilbert cube $\mathbb{I}^\omega$ is a Polish space equipped with the compact-open topology which is given by the following metric

$$\tilde{d}(f,g) = \sup_{x\in\mathbb{I}^\omega} d(f(x),g(x)) + \sup_{x\in\mathbb{I}^\omega} d(f^{-1}(x),g^{-1}(x)).$$

The important tool was also the Z-Set Unknotting Theorem (see [Ch]) saying that any two Cantor Z-sets $A, B \subseteq \mathbb{I}^\omega$ are ambiently homeomorphic, which means that there is a homeomorphism $h : \mathbb{I}^\omega \to \mathbb{I}^\omega$ such that $h[A] = B$.

Connections between cardinal coefficients of σ-ideals $\mathcal{M}$ and $\sigma\mathcal{C}_0$ are described in the following theorem.

Theorem 87 ([P8], Theorem 1.5)**.**
- $\mathrm{non}(\sigma\mathcal{C}_0) = \mathrm{non}(\mathcal{M})$,
- $\mathrm{cov}(\sigma\mathcal{C}_0) = \mathrm{cov}(\mathcal{M})$,
- $\mathrm{add}(\sigma\mathcal{C}_0) = \mathrm{add}(\mathcal{M}) = \mathrm{add}(\sigma\mathcal{C}_0, \mathcal{M})$,
- $\mathrm{cof}(\sigma\mathcal{C}_0) = \mathrm{cof}(\mathcal{M}) = \mathrm{cof}(\sigma\mathcal{C}, \mathcal{M})$.

In [P8] the smallest topologically invariant σ-ideal not included in $\mathcal{M}$ was found. It turns out that it is the σ-ideal $\sigma\mathcal{G}_0$ generated by so called *tame-G_δ sets*,

An open subset U of $\mathbb{I}^\omega$ is called a *tame open ball* if

- its closure $\bar{U}$ in $\mathbb{I}^\omega$ is homeomorphic to the Hilbert cube;
- its boundary ∂U in $\mathbb{I}^\omega$ is homeomorphic to the Hilbert cube;
- ∂U is a Z-set in $\bar{U}$ and in $\mathbb{I}^\omega \setminus U$.

By [Ch], tame open balls form a base of the topology of the Hilbert cube.

A subset U of $\mathbb{I}^\omega$ is called a *tame open set* in $\mathbb{I}^\omega$ if $U = \bigcup\mathcal{U}$ for some vanishing family $\mathcal{U}$ of tame open balls with pairwise disjoint closures in $\mathbb{I}^\omega$. The family $\mathcal{U}$ is unique and coincides with the family $\mathcal{C}(U)$ of all connected components of U. By $\bar{\mathcal{C}}(U) = \{\bar{C} : C \in \mathcal{C}(U)\}$ we shall denote the (disjoint) family of closures of all connected components of the set U.

A subset G of $\mathbb{I}^\omega$ is called a *tame G_δ-set* in $\mathbb{I}^\omega$ if $G = \bigcap_{n\in\omega} U_n$ for some sequence $(U_n)_{n\in\omega}$ of tame open sets in $\mathbb{I}^\omega$ such that $\bigcup\bar{\mathcal{C}}(U_{n+1}) \subseteq U_n$ for every $n \in \omega$ and the family $\bigcup_{n\in\omega}\bar{\mathcal{C}}(U_n)$ is vanishing in $\mathbb{I}^\omega$.

By [BR], a dense G_δ-set G in $\mathbb{I}^\omega$ is minimal (i.e. for every dense G_δ-set H there is a homeomorphism h of the Hilbert cube such that $h[G] \subseteq H$) if and only if G is a dense tame G_δ set in $\mathbb{I}^\omega$.

Theorem 88 ([P8], Theorem 1.3)**.** *The σ-ideal $\sigma\mathcal{G}_0$ is contained in each topologically invariant σ-ideal $\mathcal{I} \not\subseteq \mathcal{M}$ with BP-base on $\mathbb{I}^\omega$.*

The cardinal coefficients of the smallest topologically invariant σ-ideal with BP-base, which is not contained in the ideal $\mathcal{M}$ of meager subsets in $\mathbb{I}^\omega$ are described in the following theorem.

Theorem 89 ([P8], Theorem 1.6)**.**

$$\omega_1 \le \mathrm{add}(\sigma\mathcal{G}_0) \le \mathrm{cov}(\sigma\mathcal{G}_0) \le \mathrm{add}(\mathcal{M}) \le \mathrm{cof}(\mathcal{M}) \le \mathrm{non}(\sigma\mathcal{G}_0) \le \mathrm{cof}(\sigma\mathcal{G}_0) \le \mathfrak{c}.$$

Summarizing, the cardinal characteristics of any nontrivial topologically invariant σ-ideals $\mathcal{I} \subseteq \mathcal{M}$ and $\mathcal{J} \not\subseteq \mathcal{M}$ with analytic base on the Hilbert cube fit in the following variant of Cichoń's diagram:

Theorem 90 ([P8], Corollary 1.7)**.**

$$\begin{tikzcd}
 & & & & \mathrm{non}(\sigma\mathcal{G}_0) \arrow[r] & \mathrm{non}(\mathcal{J}) \arrow[r] & \mathrm{cof}(\mathcal{J}) \arrow[r] & \mathfrak{c} \\
 & & \mathrm{non}(\mathcal{I}) \arrow[r, equal] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cof}(\mathcal{I}) \arrow[urr, bend right] & & \\
 & & \mathrm{add}(\mathcal{I}) \arrow[r] \arrow[u] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r, equal] \arrow[u] & \mathrm{cov}(\mathcal{I}) \arrow[u] & & \\
\omega_1 \arrow[r] \arrow[urr, bend left] & \mathrm{add}(\mathcal{J}) \arrow[r] & \mathrm{cov}(\mathcal{J}) \arrow[r] & \mathrm{cov}(\sigma\mathcal{G}_0) \arrow[u] & & & &
\end{tikzcd}$$

Next, we describe certain topologically invariant σ-ideals $\mathcal{I}$ with σ-compact base on the Hilbert cube whose cardinal characteristics coincide with the respective cardinal characteristics of the ideal $\mathcal{M}$. In the following definition for a compact topological space X by $\mathcal{K}(X)$ we shall denote the family of all compact subsets of X endowed with the Vietoris topology. The hyperspace $\mathcal{K}(X)$ is partially ordered by the inclusion relation.

Definition 15. *An ideal $\mathcal{I}$ on a topological space X*

- *is a G_δ-ideal if the set $\mathcal{I} \cap \mathcal{K}(X)$ is of type G_δ in the hyperspace $\mathcal{K}(X)$;*
- *has the Solecki property $(*)$ if for any countable family $\mathcal{A} \subseteq \mathcal{I} \cap \mathcal{K}(X)$ the union $\bigcup \mathcal{A}$ is contained in a G_δ-subset $G \subseteq X$ such that $\mathcal{K}(G) \subseteq \mathcal{I}$;*
- *is a $\sigma^{(*)}$-ideal if there is a sequence $(\mathcal{I}_n)_{n\in\omega}$ of G_δ-ideals with the Solecki property $(*)$ on X such that $\mathcal{I} = \{A \in \mathcal{P}(X) : A \subseteq \bigcup \mathcal{A}$ for a countable subfamily $\mathcal{A} \subset \bigcup_{n\in\omega} \mathcal{I}_n \cap \mathcal{K}(X)\}$.*

Ideals with above properties were considered by S. Solecki in [So].

Theorem 91 ([P8], Corollary 1.10)**.** *For any nontrivial topologically invariant $\sigma^{(*)}$-ideal $\mathcal{I}$ on $\mathbb{I}^\omega$ has*

$$\mathrm{add}(\mathcal{I}) = \mathrm{add}(\mathcal{M}),\ \ \mathrm{cov}(\mathcal{I}) = \mathrm{cov}(\mathcal{M}),\ \ \mathrm{non}(\mathcal{I}) = \mathrm{non}(\mathcal{M})\ \ \textit{and}\ \ \mathrm{cof}(\mathcal{I}) = \mathrm{cof}(\mathcal{M}).$$

Using Theorem 91 we calculate the cardinal characteristics of the σ-ideals $\sigma(\mathcal{Z}_n \cap \overline{\mathcal{D}}_m)$, $n, m \le \omega$, thus answering Problem 2.6 of [BCZ]. The σ-ideal $\sigma(\mathcal{Z}_n \cap \overline{\mathcal{D}}_m)$ is generated by closed Z_n-subsets of $\mathbb{I}^\omega$ of topological dimension smaller or equal to m.

Corollary 11 ([P8], Corollary 1.11)**.** *For every $n, m \le \omega$ the σ-ideal $\mathcal{I} = \sigma(\mathcal{Z}_n \cap \overline{\mathcal{D}}_m)$ is a $\sigma^{(*)}$-ideal on $\mathbb{I}^\omega$ with cardinal characteristics*

$$\mathrm{add}(\mathcal{I}) = \mathrm{add}(\mathcal{M}),\ \ \mathrm{cov}(\mathcal{I}) = \mathrm{cov}(\mathcal{M}),\ \ \mathrm{non}(\mathcal{I}) = \mathrm{non}(\mathcal{M}),\ \ \mathrm{cof}(\mathcal{I}) = \mathrm{cof}(\mathcal{M}).$$

In paper [P9] (co-authored by T. Banakh and R. Rałowski) σ-ideals invariant under homeomorphisms preserving measure on good Cantor measure spaces were classified.

A *Cantor measure space* is a pair (X, μ) consisting of a topological space X and a σ-additive measure $\mu : \mathcal{B}(X) \to [0, \infty)$ defined on the σ-algebra of Borel subsets of X where X is a homeomorphic copy of the Cantor cube $\{0,1\}^\omega$. A Cantor measure space (X, μ) is called *good* if its measure μ is *good* in the sense of Akin [Ak], i.e., μ is continuous, *strictly positive* (which means that $\mu(U) > 0$ for any nonempty open set $U \subseteq X$), and satisfies the *Subset Condition* which means that for any clopen sets $U, V \subseteq X$ with $\mu(U) < \mu(V)$ there is a clopen set $U' \subseteq V$ such that $\mu(U') = \mu(U)$.

The class of good Cantor measure spaces includes all infinite compact metrizable zero-dimensional topological groups G endowed with the Haar measure.

Theorem 92 ([P9], Theorem 1.1)**.** *Each nontrivial invariant σ-ideal with analytic base on a good Cantor measure space (X, μ) is equal to one of the σ-ideals: $\mathcal{E}$, $\mathcal{N} \cap \mathcal{M}$, $\mathcal{N}$, $\mathcal{M}$. ($\mathcal{E}$ is the σ-ideal generated by closed μ-measure zero sets.)*

Theorem 92 is a consequence of series of lemmas describing homogeneity properties of good Cantor measure spaces.

Lemma 16 ([P9], Lemma 2.2)**.** *Let (X, μ) be a good Cantor measure space, $U \subseteq X$ be a clopen set and $K \subseteq U$ be a compact subset. For every $\alpha \in \mu[\mathrm{Clop(X)}]$ with $\mu(K) < \alpha \le \mu(U)$ there is a clopen subset $V \subseteq U$ such that $K \subseteq V$ and $\mu(V) = \alpha$.*

Lemma 17 ([P9], Lemma 2.4)**.** *Any measure-preserving homeomorphism $f : A \to B$ between closed nowhere dense subsets $A, B \subseteq X$ of a good Cantor measure space (X, μ) extends to a measure-preserving homeomorphism $f : X \to X$ of X.*

Lemma 18 ([P9], Lemma 2.5)**.** *Let (X, μ), (Y, λ) be Cantor measure spaces such that $\mu(X) < \lambda(Y)$ and the measure λ is strictly positive. Let $G_X \subseteq X$ and $G_Y \subseteq Y$ be two G_δ-sets of measure $\mu(G_X) = \lambda(G_Y) = 0$ such that G_Y is dense in Y. Then there is a measure-preserving embedding $f : X \to Y$ such that $f[G_X] \subseteq G_Y$.*

Lemma 19 ([P9], Lemma 2.6)**.** *Let (X, μ) be a good Cantor measure space, A be a closed nowhere dense subset and $B \subseteq X$ be a Borel subset of measure $\mu(B) > \mu(A)$ in X. Then there is a measure-preserving homeomorphism $h : X \to X$ such that $h[A] \subseteq B$.*

Lemma 20 ([P9], Lemma 2.9)**.** *Let (X, μ) be a good Cantor measure space and d be a metric generating the topology of X. Let $B \subseteq X$ be a Borel subset of measure $\mu(B) = \mu(X)$. For any ε, homeomorphism $f \in \mathcal{H}_\mu(X)$ and closed nowhere dense subsets $A \subseteq C$ in X with $f(A) \subseteq B$, there exists a homeomorphism $g \in \mathcal{H}_\mu(X)$ such that $g \restriction A = f \restriction A$, $g[C] \subseteq B$ and $d_{\mathcal{H}}(f, g) < \varepsilon$.*

Lemma 21 ([P9], Lemma 2.10)**.** *For any meager F_σ-sets $A, B \subseteq X$ of measure $\mu(A) = \mu(B) = \mu(X)$ in a good Cantor measure space (X, μ) there is a measure-preserving homeomorphism $h \in \mathcal{H}_\mu(X)$ such that $h[A] = B$.*

Lemma 22 ([P9], Lemma 2.11)**.** *If an analytic subset $A \subseteq X$ of a Cantor measure space (X, μ) is not contained in the σ-ideal $\mathcal{E}$, then A contains a G_δ-subset G of X such that $\mu(G) = 0$ and the measure $\mu \restriction \bar{G}$ is strictly positive.*

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