A NOTE ON ROSS'S PROOF OF LYAPUNOV'S CONVEXITY THEOREM

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Abstract. This article presents a simplified version of Ross's proof of Lyapunov's convexity theorem.

Lyapunov's convexity theorem says that the range of a finite atomless vector measure is convex. Among many proofs of this theorem, for example [1], [2] and [4], Ross's proof [3] is considered an elementary one. We think Ross's proof could be further simplified by a simple geometric observation. It is known ([3]) that proving Lyapunov's theorem can be easily reduced to the following condition for an atomless measure $\mu = (\mu_1, \dots, \mu_n)\colon \mathcal{A} \to [0, \infty)$.

$(*)$ For all $E \in \mathcal{A}$ there exists $F \subseteq E$ and $r \in (0, 1)$ such that
$$\mu(F) = r\mu(E).$$

While it is necessary to assume that the domain of a measure is a σ-algebra, condition $(*)$ leads naturally to measures defined on λ-systems. Throughout this paper, a *measure* is a σ-additive non-negative function on some λ-system, as it is appropriate for an inductive proof.

Lemma 1. *If $\mu, \nu\colon \mathcal{B} \to \mathbb{R}$ are finite measures, ν is atomless and $\mu \ll \nu$, then for any $\varepsilon > 0$ and $B \in \mathcal{B}$ with $\mu(B) > 0$, there exists $A \in \mathcal{B}$, $A \subseteq B$ such that $0 < \mu(A) < \varepsilon$, i.e. μ is atomless.*

Proof. Suppose there exists $\varepsilon > 0$ such that for every $A \in \mathcal{A}$, $A \subseteq B$ we have $\mu(A) \notin (0, \varepsilon)$. Define $C_0 := B$. Then fix $n \in \mathbb{N}$ and assume there exist sets $C_n \subseteq \ldots \subseteq C_0$ such that $\nu(C_i) = \frac{1}{2^i}\nu(B)$ and $\mu(C_i) \geqslant \varepsilon$ for all $i \leqslant n$. Since ν is atomless, there exists a partition $C_n = C \cup C'$ such that $\nu(C) = \nu(C') = \frac{1}{2}\nu(C_n)$. One of the sets C and C' has positive measure μ and we define C_{n+1} to be this set. Finally, $\mu(\bigcap_n C_n) \geqslant \varepsilon$ and $\nu(\bigcap_n C_n) = 0$, which contradicts the assumption $\mu \ll \nu$. $\square$

Lemma 2. *Let $\mu, \nu\colon \mathcal{B} \to \mathbb{R}$ be finite measures and $\varepsilon \geqslant 0$. Then for any $B \in \mathcal{B}$ there exists $A \subseteq B, A \in \mathcal{B}$ such that $\nu(A) - \mu(A) \leqslant \varepsilon$ and A is maximal with respect to this inequality and measure ν, i.e. for any $A' \in \mathcal{B}$ such that $A \subseteq A' \subseteq B$ and $\nu(A' \setminus A) > 0$ we have $\nu(A') - \mu(A') > \varepsilon$.*

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Proof. Consider the family

$$\mathcal{R} = \{A \in \mathcal{B}\colon A \subseteq B \text{ and } \nu(A) - \mu(A) \leqslant \varepsilon\}$$

partially ordered by the relation

$$A < A' \iff A \subseteq A' \text{ and } \nu(A) < \nu(A').$$

Suppose that a non-empty chain $\mathcal{L} \subseteq \mathcal{R}$ has no maximal element and let $C_n \in \mathcal{L}$ be such that $\nu(C_n) > \sup\{\nu(A)\colon A \in \mathcal{L}\} - \frac{1}{n}$ and $\nu(C_n) > \nu(C_{n-1})$. Then $\bigcup_n C_n$ is an upper bound for $\mathcal{L}$. Indeed, otherwise there is $D \in \mathcal{L}$ such that $D \not\subseteq \bigcup_n C_n$, hence $\bigcup_n C_n \subseteq D$ and $\nu(D) = \sup\{\nu(A)\colon A \in \mathcal{L}\}$; a contradiction. By Zorn's lemma there exists a maximal element in $\mathcal{R}$. $\square$

Theorem 1. *If $\mu_1, \dots, \mu_n\colon \mathcal{A} \to [0, \infty)$ are finite and atomless measures, then the vector measure $(\mu_1, \dots, \mu_n)$ satisfies condition $(*)$.*

Proof. We start with an additional assumption $\mu_1 \ll \ldots \ll \mu_n$, which can be made without loss of generality. To see this, consider $\mu' = (\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \ldots + \mu_n)$ instead of μ. The proof will be inductive with respect to n. For $n = 1$ condition $(*)$ means that a measure is atomless.

Fix $n \in \mathbb{N}$ and assume that $(*)$ holds for any n measures as in the assumptions. Let $\mu = (\mu_1, \dots, \mu_{n+1})$, where $\mu_i\colon \mathcal{A} \to \mathbb{R}$ are finite, atomless measures and $\mu_1 \ll \ldots \ll \mu_{n+1}$. Let $\nu = (\mu_2, \dots, \mu_{n+1})$. Fix $E \in \mathcal{A}$ such that $\nu(E) \neq 0 \in \mathbb{R}^n$. Consider the family

$$\mathcal{B} = \left\{B \subseteq E, B \in \mathcal{A} : \exists_{t \in [0,1]} \nu(B) = t\nu(E)\right\},$$

which is a λ-system on E. There exists a measure $\nu^*\colon \mathcal{B} \to \mathbb{R}$ such that $\nu(B) = \nu^*(B)\nu(E)$ for all $B \in \mathcal{B}$. By induction hypothesis ν^* is an atomless measure, whose range is $[0, 1]$ and also $\mu_1|_{\mathcal{B}} \ll \nu^*$. We can assume that $\mu_1(E) = 1$. If there exists a set $B \in \mathcal{B}$ such that

$$\nu^*(B) = \mu_1(B) \text{ and } 0 < \nu^*(B) < 1,$$

then condition $(*)$ is satisfied with $r = \nu^*(B)$. In search for contradiction assume otherwise. Then there exist disjoint sets $P, Q \in \mathcal{B}$ such that $\nu^*(P) = \nu^*(Q) = \frac{1}{3}$ and $\mu_1(P) > \frac{1}{3}, \mu_1(Q) < \frac{1}{3}$. By Lemma 2 with $\varepsilon = \mu_1(P) - \nu^*(P)$ there exists $Q' \subseteq Q$ such that

$$\nu^*(Q') - \mu_1(Q') \leqslant \mu_1(P) - \nu^*(P),$$

and Q' is maximal with respect to this inequality and measure ν^*, in the sense of Lemma 2. Again by Lemma 2 with $\varepsilon = \nu^*(Q) - \mu_1(Q)$ there exists $P' \subseteq P$ such that

$$\mu_1(P') - \nu^*(P') \leqslant \nu^*(Q) - \mu_1(Q),$$

and P' is maximal with respect to this inequality and measure μ_1.

Case 1. $\mu_1(P \cup Q) < \nu^*(P \cup Q) = \frac{2}{3}$. Then $\nu^*(Q \setminus Q') > 0$, because otherwise

$$\tfrac{2}{3} = \nu^*(P \cup Q) = \nu^*(P \cup Q') < \mu_1(P \cup Q') \leqslant \mu_1(P \cup Q) < \tfrac{2}{3}.$$

Since ν^* is atomless, there exists $C \in \mathcal{B}$, $C \subseteq Q \setminus Q'$ such that

$$0 < \nu^*(C) < \mu_1(P \cup Q') - \nu^*(P \cup Q').$$

Then $Q' \cup C$ contradicts the maximality of Q'.

Case 2. $\mu_1(P \cup Q) > \nu^*(P \cup Q) = \frac{2}{3}$. Then $\mu_1(P \setminus P') > 0$, because otherwise

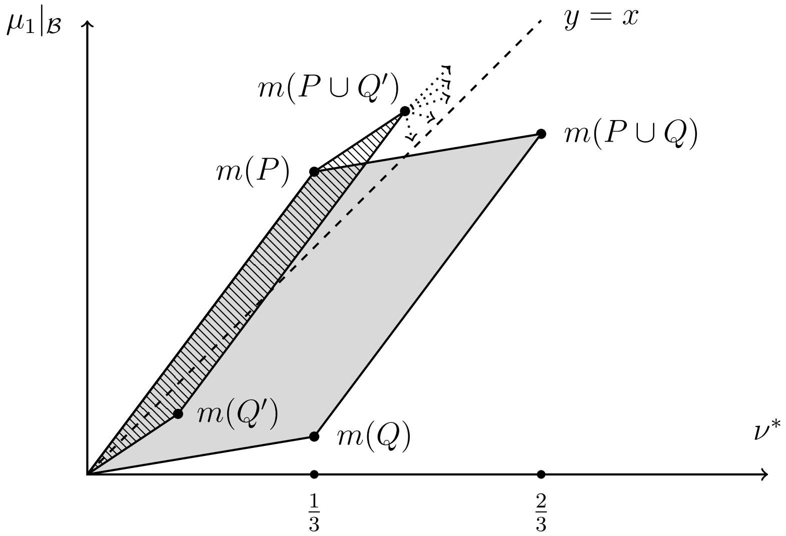
$$\tfrac{2}{3} < \mu_1(P \cup Q) = \mu_1(P' \cup Q) < \nu^*(P' \cup Q) \leqslant \nu^*(P \cup Q) = \tfrac{2}{3}.$$

By Lemma 1 there exists $C \in \mathcal{B}$, $C \subseteq P \setminus P'$ such that

$$0 < \mu_1(C) < \mu_1(P \cup Q') - \nu^*(P \cup Q').$$

Then $P' \cup C$ contradicts the maximality of P'. □

A geometric interpretation. Consider the following coordinate system, where the y-axis represents the measure $\mu_1|_{\mathcal{B}}$ and the x-axis represents the measure ν^*. Let m be the measure $(\mu_1|_{\mathcal{B}}, \nu^*)$. The area above the line $y = x$ is open, hence the maximality of Q' implies that the point $m(P \cup Q')$ should lie on the line $y = x$.



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