Katarzyna Miś, MSc
Institute of Mathematics
Faculty of Science and Technology
University of Silesia in Katowice

Katowice, 25th of February 2020

Summary of the PhD thesis

Generalized schemas and rules of reasoning in fuzzy logic

Approximate reasoning is recently an important tool since it allows us to obtain meaningful conclusions from imprecise data. It has many applications in areas like: decision theory, risk analysis, fuzzy control and data mining. In classical logic, one of the most common used rule is modus ponens (note that in the thesis we will use a notion of a scheme of reasoning) which can be seen as:

$$\frac{A \to B \ \wedge \ A}{\therefore \quad B}$$

Its generalized version can be described as follows:

RULE:	IF x is A,	THEN y is B.
FACT:	x is A'.	
CONCLUSION:		y is B'.

As we can see objects x, y have some properties A, B and A', B'. Usually, elements of a pair (A, A') are only slightly different and the same is required form a pair (B, B'). In approximate reasoning based on fuzzy sets these properties are represented by fuzzy sets. We are able to compute values of these fuzzy sets using some rules of inference - Zadeh's compositional rule of inference (*CRI*, see [9]) and Bandler-Kohout subproduct (*BKS*, see [2]). For the scheme of modus ponens, formulas that allow us to compute the conclusion's values are the following:

$$B'(y) := \sup_{x\in X} T(A'(x), I(A(x), B(y))), \quad y \in Y,$$

$$B'(y) := \inf_{x\in X} I(A'(x), T(A(x), B(y))), \quad y \in Y,$$

where T is a t-norm (or any other generalization of a classical conjunction) and I is a fuzzy implication (or a generalization of a classical implication).

One of the basic properties which is required for these rules is a property of interpolativity, which is nothing else but satisfying the classical version of modus ponens:

$$B(y) = \sup_{x\in X} T(A(x), I(A(x), B(y))), \quad y \in Y.$$

Now if we consider all possible values of fuzzy sets - all unit interval, we may obtain the following functional equations:

$$y = \sup_{x\in[0,1]} T(x, I(x, y)), \tag{CRI-GMP}$$

$$y = \inf_{x\in[0,1]} I(x, T(x, y)), \tag{BK-GMP}$$

which should be satisfied for every $y \in [0, 1]$.

In this thesis we examine generalizations of three other schemas of inference:

(i) hypothetical syllogism
$$\frac{A \to B \ \wedge \ B \to C}{\therefore \qquad A \to C}$$

(ii) modus tollens
$$\frac{A \to B \ \wedge \ \neg B}{\therefore \quad \neg A}$$

(iii) law of reduction to absurdity
$$\frac{\neg A \to B \ \wedge \ \neg B}{\therefore \quad A}$$

For these schemas of reasonig applied for two rules (CRI and BKS), we obtain the following equations:

$$I(x,y) = \sup_{z\in[0,1]} \left(T(I(x,z), I(z,y))\right), \quad x,y \in [0,1], \tag{CRI-GHS}$$

$$I_2(x,y) = \inf_{z\in[0,1]} I_1(T(x,z), T(z,y)), \quad x,y \in [0,1], \tag{BK-GHS}$$

$$N(x) = \sup_{y\in[0,1]} T(N(y), I(x,y)), \quad x \in [0,1], \tag{CRI-GMT}$$

$$N(x) = \inf_{y\in[0,1]} I(N(y), T(x,y)), \quad x \in [0,1], \tag{BK-GMT}$$

$$x = \sup_{y\in[0,1]} T(N(y), I(N(x),y)), \quad x \in [0,1], \tag{CRI-GRA}$$

$$x = \inf_{y\in[0,1]} I(N(y), T(N(x),y)), \quad x \in [0,1], \tag{BK-GRA}$$

where T is a t-norm, I, I_1, I_2 are fuzzy implications and N is a fuzzy negation. Moreover, we investigate functional inequalities which can be obtained from lattice operations in Boolean algebra and then from extention to some fuzzy connectives:

$$T(x, I(x,y)) \le y, \quad x,y \in [0,1] \tag{MP}$$

$$T(I(x,z), I(z,y)) \le I(x,y), \quad x,y,z \in [0,1] \tag{HS}$$

$$T(N(y), I(x,y)) \le N(x), \quad x,y \in [0,1] \tag{MT}$$

$$T(N(y), I(N(x),y)) \le x, \quad x,y \in [0,1], \tag{RA}$$

So far, there were investigations which concerned mainly generalized modus ponens [7], hypothetital syllogism (first research - at the end of last century, see [4]) and for other schemas - mostly for inequalities (HS), (MP), (MT), (RA) (see [8]).

Thesis is organized as follows. In Chapter 1 we recall the most important notions in the theory of fuzzy sets. In Chapter 2 we give some basic facts regarding approximate reasoning. In the main part of the thesis we consider functional equations and inequalities mentioned above when one function is given - usually a t-norm T (or a semicopula or any other generalization of a classical conjunction). Therefore we show some solutions for chosen families of fuzzy implications.

In Chapter 3 we focus on hypothetical syllogism - there are solutions for (CRI-GHS), (BK-GHS) and (HS), but also we present here some algebraic properties of a composition $\sup -T$.

Chapter 4 contains information regarding solutions of (CRI-GMP), (BK-GMP) and (MP). In Chapter 5 analogous facts for (CRI-GMT), (BK-GMT) and (MT) can be found. Solutions for (CRI-GRA), (BK-GRA) and (RA) have been described in Chapter 6.

In Chapter 7 we shortly give some remarks concerning some other possible functional equations that might be received when combinig rules of inference and different fuzzy relations.

In last Chapter 8, we present a different method of reasoning - similarity based reasoning. Also we give some remarks for two main strategies in approximate reasoning - FITA (*First Infer Then Aggregate*) and FATI (*First Aggregate Then Infer*) with respect to some proven theorems.

Some of the results presented in the thesis have been already published in referred papers in proceedings [3, 6] which were obtained in collaboration with M. Baczyński and P. Helbin and [1, 5] - prepared with M. Baczyński.

References

[1] M. Baczyński and **K. Miś**. Selected Properties of Generalized Hypothetical Syllogism Including the Case of R-implications. In J. Medina et al., editor, *Information Processing and Management of Uncertainty in Knowledge-Based Systems. Theory and Foundations*, volume 853, pages 673–684. Springer, 2018.

[2] W. Bandler and L. Kohout. Semantics of implication operators and fuzzy relational products. *Int. J. Man-Machines Studies*, pages 89–116, 1980.

[3] P. Helbin, **K. Miś**, and M. Baczyński. Some Remarks on Generalized Hypothetical Syllogism and Yager's Implication. In R. Halaš, M. Gagolewski, and R. Mesiar, editors, *New Trends in Aggregation Theory*, pages 129–139. Springer International Publishing, 2019.

[4] G. J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice Hall, New Jersey, 1995.

[5] **K. Miś** and M. Baczyński. Different Forms of Generalized Hypothetical Syllogism with Regard to R-Implications. In L. Rutkowski et al., editor, *Artificial Intelligence and Soft Computing*, volume 11508, pages 304–313. Springer, 2019.

[6] **K. Miś**, M. Baczyński, and P. Helbin. Some Remarks on the Generalized Scheme of Reduction to Absurdity and Generalized Hypothetical Syllogism in Fuzzy Logic. In *2019 Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology (EUSFLAT 2019)*, pages 423–429. Atlantis Press, 2019.

[7] E. Trillas, C. Alsina, and A. Pradera. On MPT-implication functions for fuzzy logic. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM.*, 98(1):259–271, 2004.

[8] E. Trillas, C. Alsina, and E. Renedo. On some schemes of reasoning in fuzzy logic. *New Math. Nat. Comput.*, 7(3):433–451, 2011.

[9] L. A. Zadeh. Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Trans. on Syst. Man and Cyber.*, 3:28–44, 1973.