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Katowice, May 28, 2018

Summary of PhD Thesis

Lie algebras of infinite matrices

The works of S. Lie, W. Killing and E. Cartan were the starting points for systematic development of the theory of finite-dimensional Lie algebras. We mention here the classification of finite-dimensional simple Lie algebras over the algebraically closed fields (for fields of characteristic 0 due to E. Cartan and W. Killing and for characteristic $p > 3$ given in works of R. E. Block, R. L. Wilson, H. Strade, A. Premet) and the representation theory (the highest weight classification of irreducible modules of general linear Lie algebras) [6], [8].

However, at the present time, there is no general theory of the infinite-dimensional Lie algebras. There are few classes of infinite-dimensional Lie algebras that were more or less intensively studied from the geometric point of view: the Lie algebras of vector fields, the Lie algebras of smooth mappings of a given manifold into a finite-dimensional Lie algebra, the classical Lie algebras of operators in a Hilbert or Banach space and the Kac-Moody algebras [7]. Algebraic point of view was used in investigations of free Lie algebras and graded Lie algebras. In [1] the results on the lattice structure of infinite-dimensional Lie algebras are surveyed. In many papers appear, as examples, the Lie algebra $\mathfrak{gl}_\infty$ of $\mathbb{Z} \times \mathbb{Z}$ infinite matrices over $\mathbb{C}$ which have only finite number of nonzero entries and $\mathfrak{gl}_J$ – the Lie algebra of generalized Jacobian matrices, i.e. infinite matrices having nonzero entries in a finite number of diagonals. They play important role in representation theory and physics.

We note that there is no systematic study of Lie algebras of infinite matrices. In this thesis, we consider the Lie algebra of column-finite infinite matrices indexed by positive integers $\mathbb{N}$, describe the lattice of its ideals and describe its derivations. All rings R in the thesis are commutative and with unity.

In the chapter 1 we present basic notions used in the thesis. We give descriptions of ideals and derivations of Lie algebras of finite-dimensional matrices. We recall the classification of finite-dimensional simple Lie algebras.

In the chapter 2 we survey some directions in study infinite-dimensional Lie algebras and give two fundamental examples of Lie algebras of infinite matrices – $\mathfrak{gl}_\infty$ and $\mathfrak{gl}_J$.

The third chapter contains results on Lie algebras of infinite matrices. We give the definition of the Lie algebra $\mathfrak{gl}_{cf}(\mathbb{N}, R)$ of column-finite matrices over R indexed by positive integers. We prove that $\mathfrak{gl}_{cf}(\mathbb{N}, R)$ is isomorphic with the Lie algebra of column-finite matrices indexed by integers. This shows that all results in the thesis are valid for Lie algebras of $\mathbb{Z} \times \mathbb{Z}$ column-finite matrices. We also define fundamental Lie subalgebras of $\mathfrak{gl}_{cf}(\mathbb{N}, R)$ and prove some of their properties.

The fourth chapter contains results on the Lie algebra $\mathfrak{sl}_{fr}(\mathbb{N}, R)$ of infinite matrices having nonzero entries in only finite number of rows and with trace zero. We describe its structure. For any field K, we prove the simplicity of $\mathfrak{sl}_{fr}(\mathbb{N}, K)$. We note that A. A. Baranov found classification of finitary simple Lie algebras over a field of characteristic 0 and together with H. Strade they gave classification of finitary simple Lie algebras for any algebraically closed field of prime characteristic $p > 3$ (they use the classification of simple finite-dimensional Lie algebras over an algebraically closed field of prime characteristic $p > 3$). The Lie algebra $\mathfrak{sl}_{cf}(\mathbb{N}, K)$ is a matrix representation of corresponding finitary Lie algebra, we give in [2], [4] the proof using matrix computations and which does not depend on the characteristic of the field.

In the fifth chapter we prove that every derivation of the Lie algebra of strictly upper triangular infinite matrices over R is a sum of inner and diagonal derivations. This result was published in [3]. We also prove that every derivation of $\mathfrak{gl}_{cf}(\mathbb{N}, R)$ is a sum of inner and central derivations [5].

The last chapter contains description of lattice of ideals of $\mathfrak{gl}_{cf}(\mathbb{N}, K)$. The description does not depend on the characteristic of K. As a corollary, we obtain a new uncountably dimensional simple Lie algebra [5].

References

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