

Szymon Draga, MSc  
Institute of Mathematics  
Faculty of Mathematics, Physics and Chemistry  
University of Silesia in Katowice

Katowice, February 21, 2017

## SUMMARY OF THE PHD THESIS

### SUMMABILITY AND POWER TYPE OF THE SZLENK INDEX OF DIRECT SUMS OF BANACH SPACES

The main objective of the thesis is investigation of preservation of summability and power type of the Szlenk index under taking general direct sums of Banach spaces. Moreover, we give examples of spaces with power type 1 but without summable index, and we show that summability of the Szlenk index as well as its power type are separably determined.

The notion of the Szlenk index was introduced by Wiesław Szlenk in [5] in order to show that there is no separable and reflexive Banach space that would be (isomorphically) universal for the class of all separable and reflexive Banach spaces. Later research revealed numerous connections between the Szlenk index and geometry. Among the most significant ones, one can find a result by Knaust, Odell and Schlumprecht [4], which asserts that the Szlenk index of a separable Banach space equals  $\omega$  if and only if the space may be renormed in an asymptotically uniformly smooth manner, which in turn is equivalent to the existence of a weak\*-asymptotically uniformly convex norm in the dual space. Further fundamental results concerning separable spaces with Szlenk index  $\omega$  were proved in [3] by Godefroy, Kalton and Lancien.

In the first chapter of the thesis, we give basic definitions and examples as well as we discuss connections of the Szlenk index with asymptotic geometry of Banach spaces. We determine the power type of  $\ell_p$  spaces and we also recall the definition of the original Tsirelson space, which serves as the paradigm example of a reflexive Banach space with summable Szlenk index. The results presented belong mostly to the mathematical “folklore.”

Chapter 2 is based upon the above-mentioned paper [3] and contains useful information about weak\*-null tree-maps that are a convenient tool for handling the Szlenk derivations. We also establish sharp renorming theorems for Banach spaces with Szlenk index  $\omega$ .

Chapter 3 contains results concerning summability of the Szlenk index. We start with a description of general direct sums of Banach spaces. Then we show that the  $c_0$ -sum of a family of Banach spaces with uniformly summable Szlenk index has summable Szlenk index as well. Subsequently, we prove that a similar theorem fails to hold when  $c_0$  is replaced with a more general space that has an unconditional basis and summable Szlenk index (the Tsirelson-direct sum of countably many copies of  $c_0$  is a specific example). However, the assertion remains true in the case where all summands of the direct sum are finite-dimensional. This part of the thesis is based mainly upon paper [1].

In penultimate Chapter 4, results concerning the Szlenk power type are presented. We introduce the notions of  $\ell_p$ -asymptoticity and a Banach lattice, and then we obtain certain norm estimates in such spaces. Further, we derive a formula for the power type of a general  $E$ -direct sum of a *power-type-bounded* sequence of separable Banach spaces under the hypothesis that the dual space  $E^*$  is  $\ell_p$ -asymptotic with respect to the dual basis. We also show that the hypothesis of  $\ell_p$ -asymptoticity is essential and we deduce that the Tsirelson-direct sum of countably many copies of  $c_0$  has power type 1. Here, results from [1] are also of fundamental importance.

In the final chapter, we prove that a Banach space has summable Szlenk index if and only if so has every one of its separable subspaces. We show, using analogous ideas, that a Banach space with Szlenk index  $\omega$  contains a separable subspace that has the same power type as the space itself. These results allow us to remove the separability hypothesis in the theorems from previous chapters. We finish the thesis with a result of determining the power type of “long”  $\ell_p$  spaces. Moreover, we obtain a non-trivial example of a non-separable space with summable Szlenk index—the space of all continuous functions on a Mrówka space. This chapter is based upon a fragment of paper [2] and is enriched with a few examples.

## References

- [1] S. Draga, T. Kochanek, *Direct sums and summability of the Szlenk index*, J. Funct. Anal. **271** (2016), 642–671.
- [2] S. Draga, T. Kochanek, *The Szlenk power type and tensor products of Banach spaces*, Proc. Amer. Math. Soc., **145** (2017), 1685–1698.
- [3] G. Godefroy, N.J. Kalton, G. Lancien, *Szlenk indices and uniform homeomorphisms*, Trans. Amer. Math. Soc. **353** (2001), 3895–3918.
- [4] H. Knaust, E. Odell, Th. Schlumprecht, *On asymptotic structure, the Szlenk index and UKK properties in Banach spaces*, Positivity **3** (1999), 173–199.
- [5] W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, Studia Math. **30** (1968), 53–61.