## Abstract

This dissertation presents functional inequalities connected to Sugeno integral and its applications. We focus on the Hermite-Hadmard inquality. So we start with the computer approach to solve the general form of Hermite-Hadamard inequality and to the best of our knowledge, this is the first work where a computer program may be used to solve functional inequalities.
We then study the extension of Hermite-Hadamard inequality for the case of quasi-arithmetically convex functions and it's Sugeno intergal counter part which provides a generalization and it acts as a generator for other means, in particular linear, harmonic, geometric among others and this is followed by the study of the Lagrangian mean ( non-arithmetic mean) which leads to the characterization of the logarithmic mean.
Then, on upper Hermite–Hadamard inequalities for geometric-convex and log-convex functions. This is a correction on a result by J. Sándor which is contained in article [55] where author claims, among others, that theorem 6.1 holds (cf. Theorem 2.5 in [55]).
Finally, we present the applications of fuzzy measure theory where we first propose an iterative approach to obtain the optimal value for $\lambda$ without having to solve complex polynomial functions. And then application of non-additive fuzzy measures as an alternative to traditional risk metrics like standard deviation. So we consider a Markovitz-like portfolio selection problem, where we use a fuzzy measure (a transfomation of Sugeno lambda-measure) and a d-Choquet integral to form efficient frontier. Due to the limitations of Modern Portfolio Theory (MPT) and its reliance on normal distribution assumptions, we introduce non-additive fuzzy measure, which do not assume specific probability distributions. This approach accommodates imprecision and uncertainty in financial markets, providing a more comprehensive understanding of portfolio risk. By considering diversification and asset characteristic dependencies, non-additive fuzzy measures offer a promising avenue for more accurate risk analysis and informed investment decisions.



**Keywords**: Functional equations, Hermite–Hadamard inequalities, Fuzzy measure, Sugeno integral, convex(concave) functions, Risk management, Modern Portfolio Theory, stochastic orderings, Computer assisted methods, Python.

# Bibliography


[1] **J. Aczél**, *The notion of mean values.* Norske Vid. Selsk. Forhdl., Trondhjem, 19 (1947), 83-86.

[2] **Sz. Baják and Zs. Páles,** *C*omputer aided solution of the invariance equation for two-variable Gini means, Comput. Math. Appl. **58** (2) (2009), 334–340.

[3] **Sz. Baják and Zs. Páles,** *Computer aided solution of the invariance equation for two-variable Stolarsky means,* Comput. Math. Appl. **216** (11) (2010), 3219–3227.

[4] **L. Berrone, J. Moro,** *Lagrangian means.* Aequ. Math. 55 (1998), 217–226.

[5] **M. Bessenyei and Zs. Páles,** *Higher-order generalizations of Hadamard's inequality,* Publ. Math. (Debrecen), **61** (3-4) (2002), 623–643.

[6] **M. Bessenyei and Zs. Páles,** *Characterization of higher order monotonicity via integral inequalities,* Proc. Roy. Soc. Edinburgh Sect A. **140** (4) (2010), 723–736.

[7] **G. G. Borus and A. Gilányi,** *Computer assisted solution of systems of two variable linear functional equations,* Aequat. Math. **94** (4) (2020), 723–736.

[8] **P.S. Bullen,** *Handbook of Means and Their Inequalities.* Math. Appl., Reidel, Dordrecht, 2003.

[9] **P.S. Bullen, D.S. Mitrinović, P.M. Vasić,** *Means and Their Inequalities*, Math. Appl., Reidel, Dordrecht, 1988.

[10] **Bustince, H., et al.** *Dissimilarity Based Choquet Integrals*, Information Processing and Management of Uncertainty in Knowledge-Based Systems, 1238:565–73 (2020).

[11] **J. Caballero, K. Sadarangani**, *Hermite-Hadamard inequality for fuzzy integrals.* Appl. Math. Comput. 215(2009), 2134-2138.

[12] **B.C. Carlson,** *The logarithmic mean*, Amer. Math. Monthly, 79 (1972), 615–618.

[13] **E. Castillo and A. Iglesias,** *A package for symbolic solution of real functional equations of real variables,* Aequat. Math. **54** (1997), 181–198.

[14] **G. Choquet,** *Theory of capacities,* Ann. Inst. Fourier 5 (1954) 131–296.

[15] **P. Czinder, Zs. Páles**, *An extension of the Hermite-Hadamard inequality and an application for Gini and Stolarsky means.* J. Ineq. Pure Appl. Math., 5(2) (2004), Art. 42.

[16] **M. Denuit, C. Lefevre and M. Shaked,** *The s-convex orders among real random variables, with applications,* Math. Inequal. Appl. **1** (4) (1998), 585–613.

[17] **S. S. Dragomir**, *Refinements of the Hermite-Hadamard integral inequality for log-convex functions,* The Australian Math. Soc. Gazette, submitted

[18] **S.S. Dragomir, C. Pearce,** (2003). *Selected topics on Hermite-Hadamard inequalities and applications.* Science direct working paper, 04(2003), S1574-0358.

[19] **S.S. Dragomir**, *Two refinements of Hadamard's inequalities.* Coll. of Sci. Pap. of the Fac. of Sci., Kragujevac (Yougoslavia), 11 (1990), 23–26.

[20] **S.S. Dragomir, B. Mond,** *Integral inequalities of Hadamard type for log-convex functions,* Demonstratio Math. 31 (1998), 355–364.

[21] **A. Gilányi,** *Solving linear functional equations with computer,* Math. Pannon. **9** (1) (1998), 57–70.

[22] **A. Gilányi and Zs. Páles,** *On convex functions of higher order,* Math. Inequal. Appl. **11** (2) (2008), 271–282.

[23] **P.M. Gill, C.E.M. Pearce, J. Pečarić,** *Hadamard's inequality for r-convex functions*, J. of Math. Anal. and Appl., 215 (1997), 461–470.

[24] **D. Głazowska,** *An invariance of the geometric mean with respect to lagrangian conditionally homogeneous mean-type mappings.* Demonstratio Math. 40 (2007), 289–302.

[25] **J. Hadamard,** *Étude sur les propriètès des fonctions entières et en particulier d'une fonction considerèe par Riemann.* J. Math. Pures Appl. 58, 171–215 (1893).

[26] **C. Hermite,** *Sur deux limites d'une intégrale définie.* Mathesis 3, 82 (1883) 6.

[27] **A. Ebadian, M. Oraki**, *Hermite-Hadamard inequality for Sugeno integral based on harmonically convex functions.*J. Computational Analysis and Applications, 29(3) (2021), 532-543.

[28] **A. Házy,** *Solving linear two variable functional equations with computer,* Aequat. Math. **67** (1-2) (2004), 47—62.

[29] **D. H. Hong**, *Equivalent Conditions of Hermite-Hadamard Type Inequality for Sugeno Integrals.* International Journal of Mathematical Analysis **10**(27)(2016), 1323-1331.

[30] **O. Hutnik**, *On Hadamard type inequalities for generalized weighted quasi-arithmetic means.* J. Inequal. Pure Appl. Math. 7 (2006), 3.

[31] **İ. İşcan**, *Hermite-Hadamard type inequalities for harmonically convex functions.* Hacettepe Journal of Mathematics and Statistics, 43 (6) (2014), 935-942.

[32] **M. Klaričić Bakula and K. Nikodem,** *Ohlin type theorem and its applications for strongly convex set-valued maps,* J. Convex Anal. **29** (1) (2022), 221–229.

[33] **G. Klir, B. Yuan,** *Fuzzy Sets and Fuzzy Logic: Theory and Applications,* Prentice-Hall, UK, 1995.

[34] **D.-Q. Li, X.-Q. Song, T. Yue**, *Hermite-Hadamard type inequality for Sugeno integrals.* Applied Mathematics and Computation 237(2014), 632-638.

[35] **H. Markowitz,** *Portfolio Selection,* Journal of Finance 7: 77–91 (1952).

[36] **H. Markowitz,** *Portfolio Selection: Efficient Diversification of Investments,* 2nd ed. New York: Yale University Press (1970).

[37] **H. Markowitz,***The Early History of Portfolio Theory,* Financial Analysts Journal 55: 5–16, doi:https://doi.org/10.2469/faj.v55.n4.2281 (1999).

[38] **M. Merkle and Z. D. Mitrović,** *A tight Hermite-Hadamard inequality and a generic method for comparison between residuals of inequalities with convex functions,* Period. Math. Hungar. **85** (1) (2022), 32–44.

[39] **G. Michel.** *Set functions, games and capacities in decision making,* Vol. 46. Berlin: Springer, (2016).

[40] **F. C. Mitroi, C. I. Siridon**, *Hermite-Hadamard type inequalities of convex functions with respect to a pair of Quasi-Arithmetic means.* Math.Rep., 14(64) (2012), 291-295.

[41] **J. Mrowiec, T. Rajba and Sz. Wąsowicz,** *A solution to the problem of Raşa connected with Bernstein polynomials,* J. Math. Anal. Appl. **446** (1) (2017), 864–878.

[42] **T. Nadhomi**, *Sugeno Integral for Hermite–Hadamard inequality and quasi-arithmetic means.* Annales Mathematicae Silesianae 37 (2023), no. 2, 294–305. https://doi.org/ 10.2478/amsil-2023-0007.

[43] **T. Nadhomi, M. Sablik and J. Sikorska,** *On a characterization of the logarithmic mean.* Submitted.

[44] **T. Nadhomi, C. P. Okeke, M. Sablik and T. Szostok,** *On a class of functional inequalities, a computer approach.* Submitted.

[45] **T. Nadhomi, C. P. Okeke, M. Sablik ,** *Portfolio selection based on a fuzzy measure.* Submitted.

[46] **E. Neuman,** *The weighted logarithmic mean,* J. Math. Anal. Appl., 188 (1994), 885—900.

[47] **C.P. Niculescu**, *The Hermite-Hadamard inequality for log-convex functions.* Nonlinear Anal. 75 (2012), 662–669.

[48] **C.P. Niculescu, L.-E. Persson**, *Convex Functions and their Applications.* A Contemporary Approach. CMS Books in Mathematics, vol. 23, Springer-Verlag, New York, 2006.

[49] **M. Niezgoda,** *An extension of Levin-Stečkin's theorem to uniformly convex and superquadratic functions,* Aequat. Math. **94** (2) (2020), 303–321.

[50] **C. P. Okeke, W.I. Ogala and T. Nadhomi,** *On symbolic computation of C.P. Okeke functional equations using python programming language.* Accepted. Aequat. Math.

[51] **C. P. Okeke and M. Sablik,** *Functional equation characterizing polynomial functions and an algorithm,* Results Math. **77** (3) (2022), 1–17.

[52] **A. Olbryś and T. Szostok,** *Inequalities of the Hermite-Hadamard type involving numerical differentiation formulas,* Results Math. **67** (3-4) (2015), 403–416.

[53] **T. Rajba,** *On the Ohlin lemma for Hermite-Hadamard-Fejer type inequalities,* Math. Inequal. Appl. **17** (2) (2014), 557–571.

[54] **T. Rajba,** *On a generalization of a theorem of Levin and Stečkin and inequalities of the Hermite-Hadamard type,* Math. Inequal. Appl. **20** (2) (2017), 363–375.

[55] **J. Sándor,** *On upper Hermite-Hadamard inequalities for geometric-convex and log-convex functions.* Notes on Number Theory and Discrete Mathematics, 20(5), (2014)25-30.

[56] **J. Sándor,** Corrigendum to *"On upper Hermite–Hadamard inequalities for geometric-convex and log-convex functions"* [Notes on Number Theory and Discrete Mathematics, 2014, Vol. 20, No. 5, 25–30]

[57] **M. Shaked and J.G. Shanthikumar,** *Stochastic Orders,* Springer Series in Statistics, 2007.

[58] **M. Sugeno,** *Theory of fuzzy integrals and its application,* Doctoral Thesis, Tokyo Institute of Technology, 1974.

[59] **R. Svetlozar, S. Mittnik,** *Stable Paretian Models in Finance,* Wiley, (2000) ISBN 978-0-471-95314-2.

[60] **R. Svetlozar, S. Mittnik,** *"New Approaches for Portfolio Optimization: Parting with the Bell Curve* , Risk Manager Journal (2006).

[61] **T. Szostok,** *Ohlin's lemma and some inequalities of the Hermite-Hadamard type,* Aequat. Math. **89** (2015), 915–926

[62] **T. Szostok,** *Inequalities of Hermite-Hadamard type for higher-order convex functions, revisited,* Commun. Pure Appl. Anal. **20** (2) (2021), 903–914.

[63] **D.Toker, H. Christoph, M.Stefan***Portfolio Optimization When Risk Factors Are Conditionally Varying and Heavy Tailed,* Computational Economics. 29: 333–354. doi:10.1007/s10614-006- 9071-1 (2007).

[64] **Z. Wang, G. Klir**, *Fuzzy Measure Theory.* Plenum, New York, 1992.

[65] **X.-M. Zhang, Y.-M. Chu, X.-H. Zhang**, *The Hermite-Hadamard type inequality of GA-convex functions and its application.* J. Inequal. Appl., (2010), Article ID 507560.