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Summary of PhD Thesis

On the functional equations connected to the distributivity of fuzzy implications

In classical logic conjunction distributes over disjunction and disjunction distributes over conjunction. Moreover, implication is left-distributive over conjunction and disjunction:

$$p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r),$$
$$p \to (q \vee r) \equiv (p \to q) \vee (p \to r).$$

At the same time it is neither right-distributive over conjunction nor over disjunction. However, the following two equalities, that are kind of right-distributivity of implications, hold:

$$(p \wedge q) \to r \equiv (p \to r) \vee (q \to r),$$
$$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r).$$

We can rewrite the above four classical tautologies in fuzzy logic and obtain the following distributivity equations:

$$I(x, C_1(y, z)) = C_2(I(x, y), I(x, z)), \tag{D1}$$
$$I(x, D_1(y, z)) = D_2(I(x, y), I(x, z)), \tag{D2}$$
$$I(C(x, y), z) = D(I(x, z), I(y, z)), \tag{D3}$$
$$I(D(x, y), z) = C(I(x, z), I(y, z)), \tag{D4}$$

that are satisfied for all $x, y, z \in [0, 1]$, where I is some generalization of classical implication, C, C_1, C_2 are some generalizations of classical conjunction and D, D_1, D_2 are some generalizations of classical disjunction. We can define and study those equations in any lattice $\mathcal{L} = (L, \leqslant_L)$ instead of the unit interval $[0, 1]$ with regular order „$\leqslant$" on the real line, as well.

From the functional equation's point of view J. Aczél was probably the one that studied right-distributivity first. He characterized solutions of the functional equation (D3) in the case of $C = D$, among functions I there are bounded below and functions C that are continuous, increasing, associative and have a neutral element. Part of the results presented in this thesis may be seen as a generalization of J. Aczél's theorem but with fewer assumptions on the functions F and G. As a generalization of classical implication we consider here a fuzzy implication and as a generalization of classical conjunction and disjunction - t-norms and t-conorms, respectively (or more general conjunctive and disjunctive uninorms). We study the distributivity equations (D1) - (D4) for such operators defined on different lattices with special focus on various functional equations that appear.

In the first two sections necessary fuzzy logic concepts are introduced. The background and history of studies on distributivity of fuzzy implications are outlined, as well. In Sections 3, 4 and 5 new results are presented and among them solutions to the following functional equations (with different assumptions):

$$\begin{aligned} f(m_1(x+y)) &= m_2(f(x)+f(y)), && x,y \in [0,r_1], \\ g(u_1+v_1,u_2+v_2) &= g(u_1,u_2)+g(v_1,v_2), && (u_1,u_2),(v_1,v_2) \in L^{\overline{\infty}}, \\ h(xc(y)) &= h(x)+h(xy), && x,y \in (0,\infty), \\ k(\min(j(y),1)) &= \min(k(x)+k(xy),1), && x \in [0,1], y \in (0,1], \end{aligned}$$

where:

- $f\colon [0,r_1] \to [0,r_2]$, for some constants r_1, r_2 that may be finite or infinite, and for functions m_2 that may be injective or not;
- $g\colon L^{\overline{\infty}} \to [-\infty,\infty]$, for $L^{\overline{\infty}} = \{(u_1,u_2) \in [-\infty,\infty]^2 \mid u_1 \leq u_2\}$ (function g satisfies two-dimensional Cauchy equation extended to the infinities);
- $h, c\colon (0,\infty) \to (0,\infty)$ and function h is continuous or is a bijection;
- $k\colon [0,1] \to [0,1]$, $g\colon (0,1] \to [1,\infty)$ and function k is continuous.

Most of the results in Sections 3, 4 and 5 are new and obtained by the author in collaboration with M. Baczyński, R. Ger, M. E. Kuczma or T. Szostok. Part of them have been already published either in scientific journals (see [5]) or in refereed papers in proceedings (see [4, 1, 2, 3]).

References

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