Summary of professional accomplishments

- Name and surname: **Adam Woryna**.
- Scientific degrees:
  - Master in Mathematics (Discrete Mathematics and Mathematical Foundations of Computational Sciences), Faculty of Mathematics and Physics, Silesian University of Technology in Gliwice, title of the Master Thesis: The theory of group presentations, supervisor: Professor Olga Macedońska, 03/2001,
  - Ph.D. in Mathematics, Faculty of Mathematics, Physics and Chemistry of the University of Silesia in Katowice, Title of the Ph.D. Thesis: Time-varying Mealy automata and groups generated by these automata, supervisor: Professor Vitaliy Sushchansky, 06/2005,
- University appointments: 02/2006 – present time: Adjunct (Associate Professor), Silesian University of Technology in Gliwice, The Faculty of Applied Mathematics, Section of Algebra.
- Indication of the achievement according to Article 16 Paragraph 2 of the Act of March 14, 2003 on scientific degrees and scientific title and on degrees and title in the field of art (Dz. U. 2016 r. poz. 882 ze zm. w Dz. U. z 2016 r. poz. 1311)

  The indicated scientific achievement consists of a series of eight publications entitled:

  Transducers and the topological generation of wreath products of groups.

  List of publications included in the achievement mentioned above

  [H1] A. Woryna, *The rank and generating set for iterated wreath products of cyclic groups*, Communications in Algebra, 39 (7) (2011), 2622–2631; IF 0.347,

  [H2] A. Woryna, *The rank and generating set for inverse limits of wreath products of Abelian groups*, Archiv der Mathematik, 99 (6) (2012), 557–565; IF 0.376,

  [H3] A. Woryna, *The topological decomposition of inverse limits of iterated wreath products of finite Abelian groups*, Forum Mathematicum, 25 (6) (2013), 1263-–1290; IF 0.733,

  [H4] A. Woryna, *The automaton realization of iterated wreath products of cyclic groups*, Communications in Algebra, 42 (3) (2014), 1354–1361; IF 0.388,

  [H5] A. Woryna, *On the automaton complexity of wreath powers of non-abelian finite simple groups*, Journal of Algebra, 405 (2014), 232–242; IF 0.599,

  [H6] A. Woryna, *On some universal construction of minimal topological generating sets for inverse limits of iterated wreath products of non-Abelian finite simple groups*, Journal of Algebraic Combinatorics, 42 (2) (2015), 365-–390; IF 0.874,

  [H7] A. Woryna, *The Characterization by Automata of Certain Profinite Groups*, Journal of Pure and Applied Algebra, 219 (5) (2015), 1564–1591; IF 0.669,

  [H8] A. Woryna, *On amenability of groups generated by homogeneous automorphisms and their cracks*, Forum Mathematicum, 28 (6) (2016), 1205–1213; IF 0.755.

Contents

1 Description of the field and motivation

1.1 Transducers and groups defined by them

In the classical version, a transducer (so-called Mealy type automaton) can be imagined as a directed graph with a finite set S of vertices (set of states of the automaton), in which every edge is labeled by a pair $x|y$, where x and y are elements (letters) from a fixed, finite and non-empty set X (alphabet). In this graph, any finite directed walk from an arbitrary state $s \in S$ defines in a natural way a finite sequence of pairs

$$x_1|y_1, x_2|y_2, \ldots, x_n|y_n,$$

where $x_i, y_i \in X$ for $1 \leq i \leq n$. In this walk, we say that the word $w := x_1 \ldots x_n$ with consecutive predecessors in these pairs (so-called input letters) turns into the word $v := y_1 \ldots y_n$ with consecutive successors (output letters), or that the automaton being in a state s and reading from the input tape the word w, writes on the output tape the word v. In the sequel, when speaking of "automaton", we shall mean an automaton permuting the letters, that is a graph with the property that for every vertex there are exactly $|X|$ outgoing edges from this vertex, and every letter in X belongs to the set of input letters on these outgoing edges as well as to the set of output letters. In particular, for every state $s \in S$ and a word w over X, there

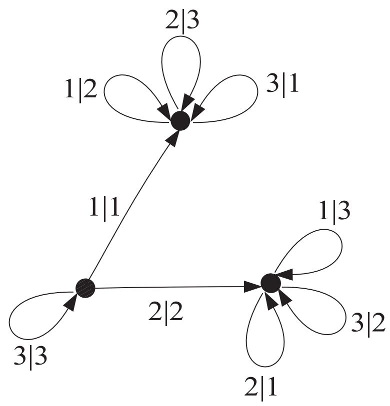

Figure 1: a minimal automaton generating an infinite 3-group

is a unique directed path starting in s such that the consecutive input letters on this path form the word w and the consecutive output letters form a word of the same length as w. Thus an arbitrary state of an automaton defines a transformation of the set X^* of all words over the alphabet X. This transformation can be described by using a so-called transition function $\varphi\colon S \times X \to S$ and an output function $\psi\colon S \times X \to X$ of the automaton, which define this automaton uniquely and describe it as a machine, which being in a state $s \in S$ and reading from the input tape a letter $x \in X$, goes to the state $\varphi(s, x)$ and writes on the output tape the letter $\psi(s, x)$. We shall denote such an automaton as a quadruple

$$A = (S, X, \varphi, \psi).$$

Then for every state $s \in S$ the image of any nonempty word $w = x_1 \ldots x_n$ under the transformation $\widetilde{s}\colon X^* \to X^*$ defined by s can be computed as follows:

$$\widetilde{s}(w) = \psi(s_1, x_1) \ldots \psi(s_n, x_n),$$

where the states $s_1, \ldots, s_n$ are defined recursively: $s_1 := s$, $s_{i+1} := \varphi(s_i, x_i)$ for $1 \leq i \leq n-1$. We also define $\widetilde{s}(\epsilon) := \epsilon$, where ϵ is the empty word (unique sequence of length zero).

The assumption that the automaton $A = (S, X, \varphi, \psi)$ permutes the alphabet implies that the transformations $\widetilde{s}$ ($s \in S$) are permutations of the set X^*, that is $\widetilde{s} \in Sym(X^*)$. It can be seen directly from the construction of these transformations that they preserve the lengths and the beginning of words, that is for any $w, v \in X^*$, we have: $|\widetilde{s}(w)| = |w|$, and if w and v have a common beginning (prefix) of a given length, then so their images $\widetilde{s}(w)$ and $\widetilde{s}(v)$.

We call a transformation $f\colon X^* \to X^*$ for which there is a Mealy automaton $A = (S, X, \varphi, \psi)$ such that $f = \widetilde{s}$ for some $s \in S$ an automaton transformation over the alphabet X. We denote the set of all automaton transformation over X by $\mathcal{MA}(X^*)$. Both the composition of automaton transformations and the inverse of an automaton transformation is also an automaton transformation. In particular $\mathcal{MA}(X^*) \leq Sym(X^*)$. Every subgroup $G \leq \mathcal{MA}(X^*)$ is called an automata group. For a single Mealy automaton $A = (S, X, \varphi, \psi)$, the group generated by the transformations $\widetilde{s} \in Sym(X^*)$ for $s \in S$ is called the group generated by the automaton A and is denoted by $G(A)$:

$$G(A) := \langle \widetilde{s}\colon s \in S \rangle.$$

Hence the group $G(A) \leq \mathcal{MA}(X^*)$ is an example of a finitely generated automata group.

The notion of an automata group was introduced by V. M. Glushkov ([28]) in 1961, where he conjectured that it is possible to obtain in this way an infinite finitely generated torsion group, that is a group solving the famous Burnside problem from 1902. It was confirmed in 1972 by S. V. Aleshin ([1]), who constructed for every prime $p \geq 2$ an infinite p-group generated by two transformations defined by two states of some two distinct Mealy automata over a p-letter alphabet, one automaton having 3 states and the second automaton having $p^2 + p + 3$ states.

An another pioneering construction of a family of infinite p-groups generated by two automaton transformations over a p-letter alphabet introduced V. Sushchansky ([75]) in 1979. He used for them the algebraic language of "tableaux" and truncated polynomials over finite fields – the method introduced by L. Kaloujnine ([48]) to study the iterated wreath products. In 2006, I. Bondarenko and D. Savchuk ([19]) investigated the sections of the Sushchansky transformations, and obtained in this way a Mealy automaton with $2p^2 + p + 5$ states. They derived various properties of the group generated by this automaton (the so-called self-similar closure of the corresponding Sushchansky p-group).

In 1980 R. I. Grigorchuk ([30]) constructed a 5-state Mealy automaton over the binary alphabet and showed that this automaton generates an infinite 2-group, which is presently called the Grigorchuk group. Also, for every prime $p \geq 3$, Grigorchuk ([35]) constructed a minimal Mealy automaton (with respect to the number of states) generating an infinite p-group. This automaton has 3 states, and it works over a p-letter alphabet (the case $p = 3$ is depicted in Fig. 1). In particular, there is no 2-state Mealy automaton over a p-letter alphabet which generates an infinite p-group. On the other hand, in the last year, I discovered for every prime $p \geq 3$ the 2-state Mealy automaton A over a p-letter alphabet which defines a universal embedding for finite p-groups, that is every finite p-group can be embedded into the group $G(A)$ generated by this automaton (the case $p = 3$ is depicted in Fig. 2). This is the only known example of a 2-state Mealy automaton which generates a branch group and one of the two known examples (apart from the Apollonian group – [36]) of a regularly branch group which is an indicable group (i.e. maps onto the infinite cyclic group). I reported this result in Kiev during the International Conference "Groups and actions: geometry and dynamics" ([84]) and at the seminar on Group Theory in the University of Geneva (Switzerland) at Grigorchuk's invitation ([85]). For $p = 2$, the existence of such an automaton excludes the known classification of

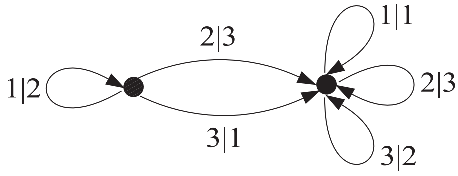

Figure 2: an automaton defining the universal embedding for finite 3-groups

groups (up to isomorphism) generated by a 2-state Mealy automaton over the binary alphabet; these are ([35]): the trivial group, the cyclic groups C_2 and C_∞, the Klein group $C_2 \times C_2$, the infinite dihedral group D_∞ and the lamplighter group $C_2 wr C_\infty$, that is the semidirect product $\bigoplus_{C_\infty} C_2 \rtimes C_\infty$ with C_∞ acting on the direct sum $\bigoplus_{C_\infty} C_2$ by left shift. Presently, there are some partial results ([17]) in classification of groups generated by a 3-state Mealy automaton over the binary alphabet (all finite and all abelian groups of this type are already classified).

The automata groups also constitute an interesting object to study the classical algorithmic problems in group theory. For example, the known construction of composition of automata and the construction of the inverse to an automaton, as well as the algorithm verifying if a given state of an automaton defines the identity map (it is enough to check the outgoing paths of length not greater than the number of all states) imply that finitely generated automata groups have solvable word problem. On the other hand, in 2012, Z. Šunić and E. Ventura ([74]) obtained the construction of a Mealy automaton A such that the conjugacy problem is not solvable in the group $G(A)$. On basis of this construction, they also proved that the isomorphism problem is not solvable in the class of groups generated by a Mealy automaton.

Undoubtedly, much of a contribution to a great interest in automata groups had the Grigorchuk group, which also solved the Milnor problem from 1968 on the possible types of group growth (the Grigorchuk group has an intermediate growth – [31]) as well as the Day problem from 1957 concerning the existence of an amenable group which is not elementary amenable ([32]). Various interesting examples of groups generated by a Mealy automaton have appeared up to the present day. They are still intensively investigated (including the Grigorchuk group) and many of them confirm one of the greater phenomenon in the modern group theory, that is an automaton itself may have a very simple structure, being equipped with only two or three states and working over an alphabet with a small number of letters, and yet it demonstrates exoticism and high complexity as for the algebraic, geometric or algorithmic properties of the group it generates.

1.2 Trees of words: regular trees and spherically homogeneous trees

The set X^* of finite words over an alphabet X has the structure of an infinite locally finite rooted tree: two words are connected with an edge if and only if one of them is obtained from the other by adding a single letter to the end. The set X^n of words of length n ($n \geq 0$) forms the n-th level of the tree X^*, that is the set of all vertices with the distance n from the root (which is the empty word ϵ). The tree X^* is called a regular rooted tree, because for every vertex $w \in X^*$ the number of its children (i.e. the words of the form wx for $x \in X$) does not depend on w and is equal to $|X|$.

It is natural to consider a wider class of locally finite rooted trees, that is the trees in which any two vertices in the same level (i.e. with the same distance from the root) have the same number of children. Every such a tree is isomorphic to the tree X^* of finite words over a changing alphabet X, which is defined as an infinite sequence

$$X := (X_1, X_2, \ldots),$$

of alphabets X_i. The words of the tree X^* constitute finite sequences of letters $x_1x_2\ldots x_n$, where $x_i \in X_i$ for $1 \leq i \leq n$ (we will not separate the letters by commas). Thus the elements of the cartesian product $X^n := X_1 \times \ldots \times X_n$ $(n \geq 1)$ are the words of length n, and these words form the n-th level of the tree X^* (we assume $X^0 := \{\epsilon\}$). In particular, the number of children of any vertex in level n $(n \geq 0)$ is equal to $|X_{n+1}|$. The first four levels of an exemplary tree X^* are depicted in Fig. 3.

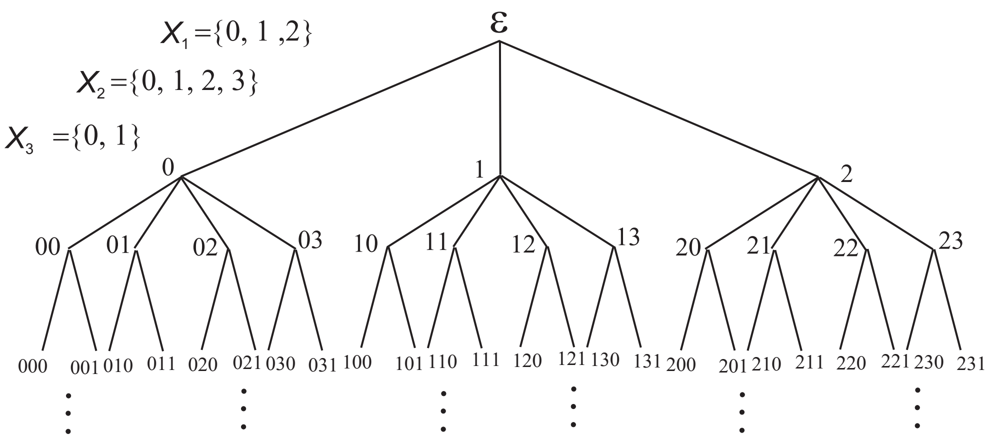

Figure 3: the exemplary tree X^*

Definition 1 We call a changing alphabet $X = (X_i)_{i\geq 1}$ bounded if the sequence $(|X_i|)_{i\geq 1}$ is bounded. Otherwise we call X unbounded. If the sequence X is constant, then it is called a fixed alphabet and identified with the set X_1.

Remark 1 In the sequel, we shall assume that if $X = (X_i)_{i\geq 1}$ is a changing alphabet, then the sets X_i are all finite and each of them has at least two elements.

1.3 The group $Aut(X^*)$ and groups defined by time-varying automata

The automorphism group $Aut(X^*)$ of the tree of words X^* over a changing alphabet $X = (X_i)_{i\geq 1}$ consists of all permutations of the vertex set which preserve the root and the vertex adjacency. These are exactly those permutations of the vertex set X^* which preserve the lengths and the beginnings of words.

In the case when the alphabet $X = (X_i)_{i\geq 1}$ is fixed, the group $\mathcal{MA}(X^*)$ of all transformations defined by Mealy-type automata over X is a proper subgroup in $Aut(X^*)$. If X is not fixed, then we can also identify the groups generated by automata among the subgroups of $Aut(X^*)$. We refer to the corresponding automata as time-varying automata, or automata over a changing alphabet. Such an automaton is created from a Mealy-type automaton by putting a discrete time-scale, which allows to change the transition and output functions in the consecutive moments of its action.

Definition 2 An automaton A over a changing alphabet $X = (X_i)_{i\geq 1}$ is defined as a finite set S od states together with two infinite sequences

$$\varphi := (\varphi_1, \varphi_2, \ldots), \quad \psi := (\psi_1, \psi_2, \ldots)$$

of transition functions $\varphi_i\colon S \times X_i \to S$ and output functions $\psi_i\colon S \times X_i \to X_i$. We denote such an automaton as a quadruple $A = (S, X, \varphi, \psi)$.

Remark 2 In the sequel, instead of the term "automaton over a changing alphabet", or "time-varying automaton", we shall use the simple term "automaton". Hence, saying "automaton", we mean the automaton from Definition 2, distinguishing this notion from a Mealy-type automaton, treated as a special case of an automaton in which the sequences $X = (X_i)_{i\geq 1}$, $\varphi = (\varphi_i)_{i\geq 1}$ and $\psi = (\psi_i)_{i\geq 1}$ are all constant (and identified with their elements).

For every state $s \in S$ of an automaton $A = (S, X, \varphi, \psi)$, we can define, analogically as for a Mealy-type automaton, a transformation $\widetilde{s}\colon X^* \to X^*$ in the following recursive way: if $w = x_1 \ldots x_n \in X^*$, then

$$\widetilde{s}(w) = \psi_1(s_1, x_1) \ldots \psi_n(s_n, x_n),$$

where $s_1 := s$ and $s_{i+1} := \varphi_i(s_i, x_i)$ for $1 \leq i \leq n-1$. It is convenient to interpret the transformation $\widetilde{s}$ as result of the action of a machine, which being in the i-th moment ($i \geq 1$) in a state $q \in S$ and reading from the input tape a letter $x \in X_i$, it goes in the next moment to the state $\varphi_i(q, x) \in S$, writes on the output tape the letter $\psi_i(q, x) \in X_i$ and continues working in the moment $i + 1$. Here, we also assume that A permutes the letters of the corresponding alphabets, that is the maps

$$\sigma_{s,i}\colon X_i \ni x \mapsto \psi_i(s, x) \in X_i, \quad i \geq 1, \quad s \in S$$

are permutations of the sets X_i. Then the automaton transformations $\widetilde{s}$ ($s \in S$) are elements of the group $Aut(X^*)$. We will refer to the map $\sigma_{s,i} \in Sym(X_i)$ ($i \geq 1$, $s \in S$) as the label of the state s in its i-th transition. If A is a Mealy automaton, then for each $s \in S$ the labels $\sigma_{s,i}$ ($i \geq 1$) coincide. In this case, we simply refer to the permutation $\sigma_s := \sigma_{s,1} \in Sym(X)$ as the label of the state s.

The set $\mathcal{TVA}(X^*)$ of all automaton transformations over the changing alphabet $X = (X_i)_{i\geq 1}$ also forms a proper subgroup in the group $Aut(X^*)$. Similarly as in the case of Mealy-type automata, we will refer to the subgroups of $\mathcal{TVA}(X^*)$ as automata groups, and for a single automaton $A = (S, X, \varphi, \psi)$, we call the group $G(A) := \langle \widetilde{s}\colon s \in S \rangle$ the group generated by the automaton A. All these groups are examples of residually finite groups, as the whole group $Aut(X^*)$ is residually finite, which follows from the observation that the n-th level stabilizer

$$Stab_{Aut(X^*)}(n) = \{g \in Aut(X^*)\colon X^n \subseteq Fix(g)\},\ n \geq 0,$$

is a normal subgroup of finite index and $\bigcap_{n\geq 0} Stab_{Aut(X^*)}(n) = \{id_{X^*}\}$.

The idea of a time-varying automaton as a tool to define and study automorphism groups of the tree of finite words over a changing alphabet was suggested by V. Sushchansky in 2001 as the theme of my Ph.D. thesis. At first, I even assumed the wider definition, allowing to change the sets of states in the discrete time-scale of an automaton (originally, I also did not assume the finiteness of these sets). I called the automata from this wider class time-varying Mealy automata. This notion previously functioned in the literature (see, for example [58]), but the investigation was reduced to the techniques for analysis and synthesis of automata, which involved only the study of their internal structure, including periodicity, representability, finding automata realizations, constructing morphisms between automata and studying the corresponding semigroups. Moreover, the previously investigated automata worked over a fixed alphabet, allowing to change solely the sets of states, the transition functions and the output functions. Consequently, that construction would not be suitable for defining automorphism groups of an arbitrary homogeneous rooted tree, but only for automorphism groups of a regular rooted tree.

1.4 Sections, vertex permutations, and the automaton transformations

Let $X = (X_i)_{i\geq 1}$ be a changing alphabet, $g \in Aut(X^*)$ and $w \in X^*$. Denote $n := |w| + 1$. Since the automorphism g preserves the lengths and the beginning of words, there is an automorphism $g_{\{w\}} \in Aut(X^*_{(n)})$ of the tree $X^*_{(n)}$ of finite words over the changing alphabet $X_{(n)} := (X_n, X_{n+1}, \ldots)$, such that

$$g(wv) = g(w)g_{\{w\}}(v),\ v \in X^*_{(n)}.$$

The automorphism $g_{\{w\}}$ is called the section of g at the word w. It describes the action of g on the subtree of X^* consisting of all descendants of the vertex w (this subtree is isomorphic to the tree $X^*_{(|w|+1)}$). In the regular case, i.e. when the alphabet X is fixed, we obviously have: $g_{\{w\}} \in Aut(X^*)$. In this case, an important class of groups constitute self-similar groups, as well as contracting groups ([60]).

Definition 3 If the alphabet X is fixed, then a group $G \leq Aut(X^*)$ is called self-similar if $g_{\{w\}} \in G$ for all $g \in G$ and $w \in X^*$. A self-similar group $G \leq Aut(X^*)$ is called contracting if there is a finite subset $S \subseteq G$ such that for every element $g \in G$ the section of g at any sufficiently long word w belongs to S (i.e. there is $n := n_g \geq 0$ such that $g_{\{w\}} \in S$ for every word $w \in X^*$ with $|w| > n$). The set S is called the nucleus of G.

The restriction of the section $g_{\{w\}} \in Aut(X^*_{(|w|+1)})$ to the set $X_{|w|+1}$ of all one-letter words is called the vertex permutation of the automorphism g at the vertex w and denoted by $\sigma_{g,w}$:

$$\sigma_{g,w} \colon X_{|w|+1} \to X_{|w|+1}, \quad \sigma_{g,w} := g_{\{w\}}|_{X_{|w|+1}}.$$

In particular $\sigma_{g,w} \in Sym(X_{|w|+1})$. The vertex permutation $\sigma_{g,w}$ tells us how g permutes the children of the vertex w (in relative to this vertex). When assign to each vertex $w \in X^*$ the vertex permutation $\sigma_{g,w} \in Sym(X_{|w|+1})$, we obtain the portrait of the automorphism g. The portrait describes this automorphism uniquely, since the image (under g) of any word $w = x_1 \ldots x_n \in X^*$ can be computed as follows

$$g(w) = \sigma_{g,w_0}(x_1)\sigma_{g,w_1}(x_2)\ldots\sigma_{g,w_{n-1}}(x_n),$$

where $w_i = x_1 \ldots x_i$ $(0 \leq i \leq n-1)$ is the prefix of length i of the word w (further, we denote by $\prec$ the relation of being a prefix). Conversely, if we choose arbitrarily the permutations $\pi_w \in Sym(X_{|w|+1})$ $(w \in X^*)$, then there is a unique $g \in Aut(X^*)$ such that $\sigma_{g,w} = \pi_w$ for every $w \in X^*$. By using the above formula, we can extend the action of an automorphism $g \in Aut(X^*)$ to the set X^ω of infinite words over the alphabet X; namely if $w = x_1x_2\ldots \in X^\omega$, then we have $g(w) = \sigma_{g,w_0}(x_1)\sigma_{g,w_1}(x_2)\ldots$.

If $g \in Aut(X^*)$ is an automaton transformation defined by a state $s \in S$ of an automaton $A = (S, X, \varphi, \psi)$, then for every word $w \in X^*$ the section $g_{\{w\}}$ is determined by the state which

A reaches after reading the word w starting from s. In particular, for every $n \geq 0$, there are at most $|S|$ different sections of g at the words $w \in X^n$. Consequently, the sequence $(\alpha_g(n))_{n\geq 0}$ is bounded, where

$$\alpha_g(n) := |\{g_{\{w\}} \colon w \in X^n\}|.$$

Conversely, if $g \in Aut(X^*)$ is such that the sequence $(\alpha_g(n))_{n\geq 0}$ is bounded, then it is possible to construct an automaton $A = (S, X, \varphi, \psi)$ such that $g = \widetilde{s}$ for some $s \in S$. It is also not difficult to construct automorphisms $g \in Aut(X^*)$ such that the sequence $(\alpha_g(n))_{n\geq 0}$ is unbounded. The characterization of the transformations defined by Mealy automata is more restricted: if the alphabet X is fixed, then an automorphism $g \in Aut(X^*)$ is defined by a state of a Mealy automaton over X if and only if the set $\{g_{\{w\}} \colon w \in X^*\}$ is finite.

1.5 Rooted and directed automorphisms. The branch groups.

The most well-known explicit constructions of finitely generated groups $G \leq Aut(X^*)$ are based on the following two types of automorphisms.

Definition 4 An automorphism $g \in Aut(X^*)$ is called rooted if $\sigma_{g,w} = id_{X_{|w|+1}}$ for every $w \neq \epsilon$.

Definition 5 Let $g \in Aut(X^*)$. If there is $u \in X^\omega$ such that g stabilizes u (i.e. $g(u) = u$) and all nontrivial vertex permutations of g are located at the vertices with the distance from u not greater than 1 (i.e. at the vertices of the form wx, where $w \prec u$ and $x \in X_{|w|+1}$), then the automorphism g is called directed and the word u is called the direction of g. If additionally, all nontrivial vertex permutations of g are located at the vertices with the distance exactly 1 from u, and there is at most one nontrivial vertex permutation in each level of X^*, then g is called 1-directed.

For example, in a 5-state Mealy automaton defining the Grigorchuk group one of the states is trivial (i.e. it defines $id_{\{0,1\}^*}$), another state is the (unique) nontrivial rooted automorphism, and the remaining three states define 1-directed automorphisms with the common direction 1^∞. Another example are the Gupta-Sidki groups constructed by N. Gupta and S. Sidki ([40]) in 1983. For every prime $p > 2$ the corresponding Gupta-Sidki group is an infinite p-group generated by a Mealy automaton with four states over the alphabet $X = \{1, 2, \ldots, p\}$; the three of these states define rooted automorphisms of the tree X^* (one of them being the trivial automorphism id, an the other two are mutually inverse automorphisms a and a^{-1}, where $\sigma_{a,\epsilon} = \sigma := (1, 2, \ldots, p)$), and the fourth state defines the directed automorphism $b \in Aut(X^*)$ with the direction $u = p^\infty$ and with the following nontrivial vertex permutations $\sigma_{b,p^i1} \colon = \sigma$ i $\sigma_{b,p^i2} \colon = \sigma^{-1}$ for every $i \geq 0$. The group $G := \langle a, b\rangle$ is indeed generated a Mealy automaton with four states (defining the above four automorphisms), which follows from the observation that for any $g \in \{id, a, a^{-1}, b\}$ and any $w \in X^*$, we have: $g_{\{w\}} \in \{id, a, a^{-1}, b\}$. The Gupta-Sidki groups (similarly as the Grigorchuk group) are still investigated (for example, in contrast to the Grigorchuk group, the growth of these groups is not known).

The rooted and directed automorphisms can be investigated as automaton transformations. However, in the case when the alphabet $X = (X_i)_{i\geq 1}$ is unbounded (and only in this case), not every directed automorphism $g \in Aut(X^*)$ is an automaton transformation. On the other hand, every 1-directed automorphism $g \in Aut(X^*)$ is an automaton transformation (regardless of the alphabet), as we have in this case $\alpha_g(n) \leq 3$ for every $n \geq 0$.

The tree X^* of finite words and the group of automorphisms $Aut(X^*)$ of such a tree, as well as the notions of rooted and directed automorphisms, can be defined for an infinite sequence

$X = (X_i)_{i \geq 1}$ of arbitrary (finite or infinite) sets (that is not necessarily when X is a changing alphabet). But then, in general, such a tree will not be locally finite and the group $Aut(X^*)$ will not be residually finite. This generalization was investigated by A. Rozhkov ([69]) in 1986, who constructed for every prime $p \geq 3$ an infinite torsion group M_p generated by two elements of order 3 and considered the tree X^* over the sequence $X := (C_7, M_2, C_7, C_7, M_3, C_7, C_7, M_5, \ldots)$. Next, in the group $Aut(X^*)$, he constructed a rooted automorphism r and a directed automorphism d such that the group $G = \langle r, d \rangle$ is torsion and contains elements of all possible finite orders (that is for every $n \geq 1$ there is $g \in G$ such that $o(g) = n$). The question on the existence of such a group in the case when X is a changing alphabet is open. In particular, it would be interesting to find an explicit construction of an automaton over a changing alphabet generating such a group, or to show that such an automaton does not exist. In the "existence" case the corresponding alphabet would have to be unbounded.

The concept of rooted and directed automorphisms provides also the general construction of finitely generated branch groups ([6]).

Definition 6 A group $G \leq Aut(X^*)$ acting transitively on each level of the tree X^* (we say then that the action is spherically transitive) is called a branch group if for every $n \geq 0$ the subgroup $Rist_G(n) := \langle Rist_G(w) \colon w \in X^n \rangle$ is of finite index in G, where $Rist_G(w)$ consists of those elements $g \in G$ which act trivially on every word not beginning with w (a so-called rigid vertex stabilizer of w). If the groups $Rist_G(n)$ $(n \geq 0)$ are only nontrivial, then G is called weakly branch.

This important and presently intensively studied class of groups was introduced by Grigorchuk in 1997, providing a natural method for defining infinite groups whose every proper quotient is finite (i.e. just-infinite groups). It can be shown that every infinite finitely generated group can be mapped onto a just-infinite group, and the class of all finitely generated just-infinite groups naturally splits into three subclasses, one of which consists of branch groups ([33]). Initially, Grigorchuk supposed that every finitely generated branch group must be just-infinite, and the only way to construct such a group is by using (as generators) the rooted and directed automorphisms ([22]).

In the regular case, in the class of self-similar groups, we identify the subclass of regularly branch groups.

Definition 7 Let $G \leq Aut(X^*)$ be a self-similar group acting spherically transitively on a regular rooted tree X^* and let $K \triangleleft G$ be a normal subgroup of G. The group G is called regularly branch over K if the index $[G : K]$ is finite and for all $k \in K$ and $x_0 \in X$ there is $h \in K \cap Stab_G(1)$ such that $h_{\{x_0\}} = k$ and $h_{\{x\}} = id_{X^*}$ for any $x \neq x_0$. In the case when K is merely nontrivial, we say that G is regularly weakly branch.

The flagship examples of regularly branch groups are the Grigorchuk group, the Gupta-Sidki groups, as well as the first examples of groups having a non-uniformly exponential growth (i.e. the groups solving the Gromov problem) constructed by J. S. Wilsona ([81, 82]) in 2004. The famous and intensively studied examples of regularly weakly branch groups are the following groups generated by a Mealy automaton: basilica group ([38]), Bartholdi-Grigorchuk group, Brunner-Sidki-Vieira group ([6]), and the previously mentioned self-similar closures of the Sushchansky p-groups.

1.6 Iterated wreath products. The group $Aut(X^*)$ as a profinite group

Let (G, X) and (H, Y) be permutation groups of the sets X and Y, let H^X be the direct power of copies of H indexed by X with the elements written as functions $f\colon X \to H$. For every $f \in H^X$ and every $g \in G$ the pair (f, g) defines a permutation of the cartesian product $X \times Y$ of the sets X and Y in the following way:

$$(f, g)((x, y)) = (g(x), h(y)), \quad (x, y) \in X \times Y,$$

where $h := f(x) \in H$. The set of all such permutations (i.e. permutations of $X \times Y$ corresponding to all pairs $(f, g) \in H^X \times G$) forms a permutation group on $X \times Y$ which is called the permutational wreath product of the groups (G, X) and (H, Y) and is denoted by $H \wr_X G$. The use of the G-set X in the above notation is justified, as the structure of the permutational wreath product $H \wr_X G$ depends on this set. For example, if G, X and H are finite, then the group $H \wr_X G$ is also finite, and its order is equal to $|H|^{|X|} \cdot |G|$. The multiplication in this group can be described as follows

$$(f, g)(f', g') = (f \circ f'_g, g \circ g'),$$

where $f'_g = g \circ f' \in H^X$ (we use here the right action convention for composition of mappings; in particular, we have $g \circ g'(x) = g'(g(x))$ and $g \circ f(x) = f(g(x)) \in H$ for any $g, g' \in G$, $f \in H^X$ and $x \in X$). This reveals the permutational wreath product $H \wr_X G$ as a group isomorphic to the semidirect product $H^X \rtimes G$, where G acts on the direct power H^X by permuting the direct factors in the same manner as it permutes the elements of X. In particular, if X is a finite set with a fixed ordering of elements, for example $X := \{1, \ldots, m\}$ for some $m \geq 1$, then every function $f \in H^X$ is a sequence $(f(1), \ldots, f(m))$ of elements from H and the group H^X is the cartesian m-th power $H^m = H \times \ldots \times H$. Thus the elements of $H \wr_X G$ can be written in the form $(h_1, \ldots, h_m)\pi$, where $h_j \in H$ $(1 \leq j \leq m)$, $\pi \in G$. The multiplication in such a wreath product $H \wr_X G$ can be described as follows:

$$(h_1, \ldots, h_m)\pi \; (h'_1, \ldots, h'_m)\pi' = (h_1 \circ h'_{\pi(1)}, \ldots, h_m \circ h'_{\pi(m)})\pi \circ \pi'.$$

This construction is associative, i.e. if (K, Z) is a permutation group on the set Z, then both the permutational wreath product $(K \wr_Y H) \wr_X G$ and the permutational wreath product $K \wr_{X \times Y} (H \wr_X G)$ are precisely the same permutation group on the set $X \times Y \times Z$ ([57]).

Let now $(G_i, X_i)_{i \geq 1}$ be an infinite sequence of permutation groups. For every $i \geq 1$, we define the iterated permutational wreath product $W_i = \wr_{k=1}^{i} G_k$ of the first i groups as a permutation group on the cartesian product $X^i = X_1 \times \ldots \times X_i$ as follows:

$$W_1 := G_1, \quad W_{i+1} := G_{i+1} \wr_{X^{(i)}} W_i, \quad i \geq 1.$$

For every $i \in \mathbb{N}$ the mapping $(f, g) \mapsto g$, where $f \in (G_{i+1})^{X^i}$, $g \in W_i$, defines a homomorphism $\phi_i\colon W_{i+1} \to W_i$, and the sequence $(W_i, \phi_i)_{i \geq 1}$ forms an inverse system. We call the inverse limit

$$W_\infty = \wr_{i=1}^{\infty} G_i := \varprojlim_i W_i = \varprojlim_i \wr_{k=1}^{i} G_k$$

of such an inverse system the infinitely iterated permutational wreath product of the sequence $(G_i, X_i)_{i \geq 1}$. Thus, according to the definition of the inverse limit, the group W_∞ consists of all sequences $(h_i)_{i \geq 1}$ from the infinite cartesian product $\prod_{i \geq 1} W_i$ of the groups $W_i = \wr_{k=1}^{i} G_k$ which satisfy the following condition: $\phi_i(h_{i+1}) = h_i$ for each $i \geq 1$. If the sets X_i are all finite, then each W_i is a finite group and we obtain W_∞ as the inverse limit of finite groups, that is as a profinite group.

Let $X = (X_i)_{i\geq 1}$ be a changing alphabet and denote $X_1 := \{x_{1,1}, \ldots, x_{1,m_1}\}$. An arbitrary automorphism $g \in Aut(X^*)$ is uniquely described by its one-letter sections $g_{\{x_{1,1}\}}, \ldots, g_{\{x_{1,m_1}\}}$ together with the vertex permutation $\sigma_{g,\epsilon} \in Sym(X_1)$ at the root. The map

$$g \mapsto (g_{\{x_{1,1}\}}, \ldots, g_{\{x_{1,m_1}\}})\sigma_{g,\epsilon}$$

defines an isomorphism of the group $Aut(X^*)$ with the permutational wreath product

$$Aut(X^*_{(2)}) \wr_{X_1} Sym(X_1) = Aut(X^*_{(2)})^{m_1} \rtimes Sym(X_1).$$

By continuing this reasoning for the groups $Aut(X^*_{(n)})$ $(n = 2, 3, \ldots)$, we obtain for every $n \geq 1$ the isomorphism of the group $Aut(X^*)$ with the permutational wreath product $Aut(X^*_{(n+1)}) \wr_{X^n} (\wr_{i=1}^n Sym(X_i))$. In particular, the quotient group $Aut(X^*)/Stab_{Aut(X^*)}(n)$ is isomorphic to the n-iterated wreath product $\wr_{i=1}^n Sym(X_i)$. The restriction $g|^n := g|_{X^n}$ of any automorphism $g \in Aut(X^*)$ to the set $X^n = X_1 \times \ldots \times X_n$ belongs to this wreath product, and the map $g \mapsto (g|^n)_{n\geq 1}$ defines an isomorphism of the group $Aut(X^*)$ with the infinitely iterated wreath product $\wr_{i=1}^\infty Sym(X_i)$.

The above description of the group $Aut(X^*)$, as a profinite group, defines on this group a natural profinite topology, in which the stabilizers $Stab_{Aut(X^*)}(n)$ $(n \geq 0)$ of consecutive levels of the tree X^* form a basis for the neighborhoods of id_{X^*}. This topology (called the congruence topology) coincides with the metric topology in which two automorphisms are *close to each other* if for some *large* number n, they act in the same way on level n. Such a metric can be defined, for example, as follows:

$$\delta(g, h) := \inf\{(1/2)^n \colon \forall\, w \in X^n\; g(w) = h(w)\}.$$

In particular, the wreath product $\wr_{i=1}^\infty G_i$ is a closed subgroup of $Aut(X^*)$, which consists of all automorphisms such that their vertex permutations at the vertices in level n $(n \geq 0)$ of the tree X^* belong to the group G_{n+1}:

$$\wr_{i=1}^\infty G_i = \{g \in Aut(X^*) \colon \forall w \in X^*\; \sigma_{g,w} \in G_{|w|+1}\}.$$

The relation

$$g = (g_{\{x_{1,1}\}}, \ldots, g_{\{x_{1,m_1}\}})\sigma_{g,\epsilon}$$

identifies an automorphism $g \in Aut(X^*)$ with the corresponding element of the wreath product $Aut(X^*_{(2)}) \wr_{X_1} Sym(X_1)$. This relation is called the wreath recursion of the automorphism g. The multiplication of wreath recursions and the inverse operation agrees with the multiplication in the above permutational wreath product; that is, we have:

$$g^{-1} = ((g_{\{y_{1,1}\}})^{-1}, \ldots, (g_{\{y_{1,m_1}\}})^{-1})\sigma^{-1}_{g,\epsilon}, \tag{1}$$

where $y_{1,i} := \sigma^{-1}_{g,\epsilon}(x_{1,i})$ for $1 \leq i \leq m_1$, and if $h = (h_{\{x_{1,1}\}}, \ldots, h_{\{x_{1,m_1}\}})\sigma_{h,\epsilon}$, then

$$g \circ h = (g_{\{x_{1,1}\}} \circ h_{\{z_{1,1}\}}, \ldots, g_{\{x_{1,m_1}\}} \circ h_{\{z_{1,m_1}\}})\sigma_{g,\epsilon} \circ \sigma_{h,\epsilon}, \tag{2}$$

where $z_{1,i} := \sigma_{g,\epsilon}(x_{1,i})$ for $1 \leq i \leq m_1$.

Remark 3 If the vertex permutation $\sigma_{g,\epsilon}$ is trivial, then it is omitted in the wreath recursion, and we write $g = (g_{\{x_{1,1}\}}, \ldots, g_{\{x_{1,m_1}\}})$. If all the sections $g_{\{x_{1,r}\}}$ $(1 \leq r \leq m_1)$ are trivial, then g is identified with the vertex permutation $\sigma_{g,\epsilon}$. In this way, both the direct power $Aut(X^*_{(2)})^{X_1}$ and the symmetric group $Sym(X_1)$ are identified with the corresponding subgroups of the group $Aut(X^*)$ (i.e. with the stabilizer of the first level $Stab_{Aut(X^*)}(1)$ and the subgroup of rooted automorphisms of the tree X^*, respectively).

The study of infinitely iterated permutational wreath products $\wr_{i=1}^{\infty} G_i$ of finite permutation groups was initiated by L. Kaloujnine [47, 48] in the mid-40s last century. This was continued by his students Y. V. Bodnarchuk ([14]), I. D. Ivanyuta ([42]), V. Sushchansky ([26, 76, 77, 78]) and others. It turns out that every pro-p Sylow subgroup of the group $Aut(X^*)$ is of this form. The wreath products $\wr_{i=1}^{\infty} G_i$ describe profinite completions of some finitely generated branch groups and provide interesting examples and counterexamples in the theory of profinite groups ([34]). When the sequence $(G_i, X_i)_{i\geq 1}$ is constant, we obtain the infinite wreath power $\wr_{i=1}^{\infty} G^{(i)}$ of the group $G := G_1$. These wreath powers characterize the so-called self-similar groups of finite type described by a pattern of depth one ([18, 34]). Iterated permutational wreath products of finite groups appear also as symmetry groups of such combinatorial structures as nested designs ([3, 4]), or in chemistry, they describe symmetries of certain non-rigid molecules ([5, 83]). Nowadays, these groups are even found to be useful as descriptors for processing information in the human visual system ([51]).

1.7 The topological generation in the group $Aut(X^*)$

Let $X = (X_i)_{i\geq 1}$ be a changing alphabet and $G \leq Aut(X^*)$. We say that a subset $S \subseteq Aut(X^*)$ topologically generates G if the group $\langle S \rangle$ generated by S is a dense subgroup of G. By $d(G)$, we denote the rank of G, that is the minimal number of elements in a generating set (in the case when G is closed, we mean the topological rank, i.e. the minimal number of elements in a topological generating set). If $d(G) < \infty$, then we say that G is finitely generated (resp. topologically finitely generated).

Definition 8 A generating set S of a group $G \leq Aut(X^*)$ (topologically generating set in the case when G s closed) which satisfies $|S| = d(G)$ is called a minimal generating set (resp. a minimal topologically generating set).

According to the above defined profinite topology on $Aut(X^*)$, a subset S of a group $G \leq Aut(X^*)$ topologically generates this group if and only if

$$\langle s|^i \colon s \in S\rangle = \langle g|^i \colon g \in G\rangle$$

for every $i \geq 0$, where $g|^i$ denotes the restriction of g to the i-th level of the tree X^*. In particular, we have $d(G) \geq d(\overline{G})$, where $\overline{G}$ is the topological closure of G in $Aut(X^*)$.

The whole group $Aut(X^*)$ is not topologically finitely generated as the infinite direct power $C_2^{\{0,1,2,\ldots\}}$ of the cyclic group C_2 (and hence every finite power C_2^t, $t = 1, 2, \ldots$) is a homomorphic image of G. This image can be seen when assigning to each $g \in Aut(X^*)$ the sequence from $C_2^{\mathbb{N}_0}$ such that the n-th element $(n \geq 0)$ of this sequence is equal to 0 or to 1, depending on the parity of the product of the vertex permutations $\sigma_{g,w} \in Sym(X_{n+1})$ at the vertices $w \in X^n$.

In 1994, M. Bhattacharjee ([12]), when studying the wreath products $\wr_{i=1}^{\infty} Alt(n_i)$ of alternating groups of degree $n_i \geq 5$, showed that they are topologically 2-generated. Consequently, if

$|X_i| \geq 5$ for every $i \geq 1$, then the group $Aut_e(X^*) \leq Aut(X^*)$ of the alternating automorphisms (i.e. the automorphisms with the vertex permutations all even) is topologically 2-generated.

Theorem 1 (Bhattacharjee, [12]) *If $|X_i| \geq 5$ for every $i \geq 1$, then the wreath product $\wr_{i=1}^{\infty} Alt(X_i)$ is topologically 2-generated.*

In 2006, M. Quick ([67]) extended this result to arbitrary non-abelian finite simple groups.

Theorem 2 (Quick, [67]) *If $(H_i, X_i)_{i\geq 1}$ is an arbitrary sequence of non-abelian finite simple and transitive permutation groups, then the wreath product $\wr_{i=1}^{\infty} H_i$ is topologically 2-generated.*

The Quick's paper does not provide any construction of the corresponding topological generating set. In the Bhattacharjee's work, we can find such a set – it is based on some specific generators of the groups $Alt(n_i)$, which depend on the divisibility of the degree n_i by four. However, this construction seems to be quite complicated and not accessible for the further study of the group generated by the constructed set.

In 2010, Bondarenko ([15]) formulated a condition when the wreath product $\wr_{i=1}^{\infty} G_i$ of transitive permutation groups (G_i, X_i) with the uniformly bounded ranks $d(G_i)$ is topologically finitely generated.

Theorem 3 (Bonadrenko, [15]) *Let $(G_i, X_i)_{i\geq 1}$ be a sequence of transitive permutation groups. If the sequence $(d(G_i))_{i\geq 1}$ is bounded, then $d(\wr_{i=1}^{\infty} G_i) < \infty$ if and only if $d(\prod_{i\geq 1} G_i/G_i') < \infty$.*

In Theorem 3 the abelianizations $A_i := G_i/G_i'$ $(i \geq 1)$ are finite abelian groups, and hence the infinite cartesian product $\prod_{i\geq 1} A_i$ is a profinite abelian group. This profinite group can be identified with a closed subgroup of the group $Aut(X^*)$, where the changing alphabet $X = (X_i)_{i\geq 1}$ comes from regular actions of the groups A_i on themselves, that is $X_i := A_i$ for every $i \geq 1$. Then $\prod_{i\geq 1} A_i$ is a closed subgroup of the wreath product $\wr_{i=1}^{\infty} A_i$, consisting of the so-called homogeneous automorphisms, that is the automorphisms for which the vertex permutations at the vertices in any given level of X^* coincide (obviously, the vertex permutations at the vertices from distinct levels may differ). The rank of such a cartesian product can be computed as follows:

$$d(\prod_{i\geq 1} A_i) = \sup_{p\in\mathbb{P}}(\sup_{i\geq 1} \rho_{i,p}),$$

where $\mathbb{P}$ is the set of all primes, and $\rho_{i,p}$ is the number of cyclic p-groups in the canonical decomposition of the product $A_1 \times \ldots \times A_i$ into the direct product of cyclic groups of prime-power orders.

In one direction Theorem 3 is obvious because the direct product $\prod_{i\geq 1} G_i/G_i'$ is a homomorphic image of the group $\wr_{i=1}^{\infty} G_i$ (as its abelianization). For the converse, Bondarenko showed that if the groups (G_i, X_i) satisfy some additional conditions, then there exists a finite topological generating set of the wreath product $\wr_{i=1}^{\infty} G_i$, which consists of rooted and directed automorphisms. He also observed that the boundedness of the sequence $(d(G_i))_{i\geq 1}$ is not necessary for the group $\wr_{i=1}^{\infty} G_i$ to be topologically finitely generated.

A quite different and purely algebraical approach was used by E. Detomi and A. Luchcini ([23]) in 2013 to provide the following complete characterization for the wreath products $\wr_{i=1}^{\infty} G_i$ to be topologically finitely generated.

Theorem 4 (Detomi, Luchcini, [23]) *Let $(G_i, X_i)_{i\geq 1}$ be a sequence of transitive permutation groups. Then $d(\wr_{i=1}^{\infty} G_i) < \infty$ if and only if $d(\prod_{i\geq 1} G_i/G_i') < \infty$ and the sequence $(d(G_i)/N_{i-1})_{i\geq 2}$ is bounded, where $N_i := |X_1| \cdot \ldots \cdot |X_i|$ for $i \geq 1$.*

In the proof of Theorem 4, the authors investigated the factors of the maximal normal sequence (chief factors) of a finite group H and compared these factors with the chief factors of the permutational wreath product $H \wr_Y G$ with a transitive permutation group (G, Y). However, this approach, similarly as the Quick's work, does not provide any construction of a topological generating set for the group $\wr_{i=1}^{\infty} G_i$.

1.8 The previous constructions of topological generating sets for wreath products

As a part of my PhD Thesis, I showed (the paper [D5] from 2006 – the description is given on p. 48) that the wreath product $\wr_{i=1}^{\infty} C_{m_i}$ of finite cyclic transitive permutation groups C_{m_i} with a pairwise coprime orders is topologically 2-generated. But, in contrast to the above cited works, I constructed a clear and simple 2-element topological generating set for this wreath product, as well as I investigated the group generated by this set. I realized that construction by an automaton with a 2-element set of states over the alphabet $X = (X_i)_{i \geq 1}$ in which $|X_i| = m_i$ for $i \geq 1$. In particular, I introduced the wreath product $\wr_{i=1}^{\infty} C_{m_i}$ as a group generated by a 2-state automaton, that is by a minimal automaton for this wreath product.

Definition 9 We say that a group $G \leq Aut(X^*)$ is generated by an automaton $A = (S, X, \varphi, \psi)$, if the equality $G(A) = G$ holds, or, if G is closed, the equality $\overline{G(A)} = G$. Additionally, if $|S| = d(G)$, then we say that the automaton A is minimal for G, and if $|S| = d(G) + 1$ and one of the states is trivial (i.e. it defines the identity automorphism), then we call A an almost minimal automaton for G.

Definition 10 An automaton A generating a group $G \leq Aut(X^*)$ is called optimal for G, if the number of its states is not greater than the number of any other automaton generating G.

The previously known constructions of finite topological generating sets for permutational wreath products of the form $\wr_{i=1}^{\infty} G_i$ were studied only for some particular sequences $(G_i, X_i)_{i \geq 1}$ of non-abelian simple groups (such as the alternating groups $Alt(n_i)$ and the projective special linear groups $PSL_2(p_i)$), or for some unspecified perfect groups satisfying additional conditions for the actions on the sets X_i (a group G is called perfect if it coincides with the commutator subgroup $G' = [G, G]$). In particular, there was no known construction of a finite topological generating set for the wreath product $\wr_{i=1}^{\infty} A_i$ of abelian groups. All these constructions were based on rooted and directed automorphisms, which implies that the resulting sets were far from minimal (the only exceptions were two specific constructions for the wreath products of alternating groups, that is the mentioned above Bhattacharjee's construction and the listed below Wilson's construction of two directed automorphisms in the wreath product of the alternating groups $Alt(n_i)$ of degree $n_i \geq 29$). Furthermore, the notion of an automaton was not used for the description of these sets and the problem of the existence of a minimal or almost minimal automaton was not investigated. On the other hand, it was proved that the groups generated by these sets, apart from that they are dense in the corresponding wreath product, satisfy another properties. This allowed to discover interesting properties of finitely generated residually finite groups and to settle some conjectures on these groups.

The pioneer result of this type was the P. M. Neumann's ([63]) construction of a finitely generated just-infinite perfect group G isomorphic to the permutational wreath product $G \wr Alt(6)$. In his work, Neumann showed that every subnormal subgroup of G is isomorphic to the finite direct power of G, but G does not satisfy the ascending chain condition on subnormal subgroups. This provided new examples of atomic (minimal) groups with respect to the relation of *largeness* defined by S. J. Pride'a in the class of all groups.

D. Segal ([70]) constructed a group generated by a 4-element topological generating set (two rooted and two 1-directed automorphisms) of the wreath product $\wr_{i=1}^{\infty} PSL_2(p_i)$ with a suitable chosen sequence $(p_i)_{i\geq 1}$ of distinct primes, and invalidated in this way the conjecture of Lubotzky, Pyber and Shalev on subgroup growth of a finitely generated group. In the same work, he also showed, on basis of rooted and 1-directed automorphisms, that if (G_i, X_i) are non-abelian simple transitive groups satisfying: $Stab_{G_i}(x) \neq Stab_{G_i}(y)$ for $x \neq y$, then the group $\wr_{i=1}^{\infty} G_i$ contains a 63-generated just-infinite subgroup G with the congruence subgroup property (i.e. every finite index subgroup $K \leq G$ contains a stabilizer $Stab_G(n)$ for some $n = n_K \geq 1$). As a result, he obtained that all finite images of G coincide with the finite images of finitely iterated wreath products $\wr_{i=1}^{n} G_i$ $(n \geq 1)$, and that the profinite completion $\widehat{G}$ is isomorphic to the topological closure $\overline{G} = \wr_{i=1}^{\infty} G_i$. He concluded that if $\mathcal{C}$ is any non-empty collection of non-abelian finite simple groups, then there exists a 63-generated just-infinite group whose upper composition factors comprise exactly the set $\mathcal{C}$ (an upper composition factor means a composition factor of some finite image of the group).

J. S. Wilson ([81, 82]) investigated some specific generating pairs (eligible pairs) of the alternating groups $Alt(m)$ of degree $m \geq 29$. He used these pairs in the construction of two directed automorphisms of orders 2 and 3, and showed that these automorphisms generate a perfect group G isomorphic to the wreath product $G \wr Alt(m)$. This allowed him to solve the previously mentioned Gromov problem. The eligible pairs were also used by J. Brieussel ([20]) in the construction of two directed automorphisms of orders 2 and 3 in the wreath product $\wr_{i=1}^{\infty} Alt(n_i)$ of the alternating groups of degree $n_i \geq 29$, obtaining in this way a 2-generated group of intermediate growth, which is dense in this wreath product.

Some more general and geometric construction was introduced by Bondarenko ([15]). He investigated an arbitrary perfect transitive permutation group (H, X) which does not act freely on the set X. Based on an arbitrary generating set $\{h_1, \ldots, h_m\}$ of H, he defines for every $1 \leq j \leq m$ two automorphisms $r_j, d_j \in \wr_{i=1}^{\infty} H^{(i)}$ by their vertex permutations $\sigma_{r_j,w}$, $\sigma_{d_j,w}$ $(w \in X^*)$ as follows:

$$\sigma_{r_j,w} := \begin{cases} h_j, & w = \epsilon, \\ id_X, & w \neq \epsilon, \end{cases} \qquad \sigma_{d_j,w} := \begin{cases} h_j, & w = x_1^i x_2,\ i \geq 0, \\ id_X, & \text{otherwise}, \end{cases}$$

where $x_1, x_2 \in X$ are arbitrarily chosen letters with the distinct stabilizers (hence the assumption that H does not act freely). In particular r_j is a rooted automorphism and d_j is a 1-directed automorphism with the direction x_1^{∞}. He showed that the set $S = \{r_1, \ldots, r_m, d_1, \ldots, d_m\}$ topologically generates the infinite wreath power $\wr_{i=1}^{\infty} H^{(i)}$, and that $G := \langle S \rangle$ is a just-infinite regularly branch group over itself. Moreover, G has the congruence subgroup property and it contains maximal subgroups of infinite index. This is the only known example of branch groups having maximal subgroups of infinite index (for example, in the Grigorchuk group, every maximal subgroup is of index two).

2 Discussion of the results on the basis of the works [H1]–[H8]

The previously described results on finite topological generation of wreath products $\wr_{i=1}^{\infty} G_i$ have inspired me to the further (already after my PhD study) exploration of this topic. For this purpose, I used the geometric description of the group $Aut(X^*)$, basing on the analysis of the portraits of suitable constructed automorphisms, as well as the combinatorial language of automata, relying on the notion of wreath recursion. It allowed me to obtain transparent, simple, and at the same time quite universal constructions of finite topological generating sets for these wreath products. In particular, I obtained the results in the following topics:

- constructing minimal and almost minimal automata for wreath products of the form $\wr_{i=1}^{\infty} G_i$,
- the study of algebraic and geometric properties of groups generated by the constructed sets/automata,
- characterization of those sequences $(G_i, X_i)_{i\geq 1}$ for which the wreath product $\wr_{i=1}^{\infty} G_i$ is generated by a single automaton over the alphabet $X = (X_i)_{i\geq 1}$.

The papers **[H1]–[H8]** contain the solutions of the following problems:

- construction of a minimal Mealy automaton for the infinite wreath power of a nontrivial group (in particular, for the infinite wreath power of an arbitrary alternating group $Alt(n)$ of degree $n \geq 5$) (**[H5]**),
- obtaining a universal almost minimal automaton realization for wreath products $\wr_{i=1}^{\infty} H_i$ of finite non-abelian simple groups; the study of algebraic and geometric properties of the group generated by the constructed automaton (**[H6]**),
- computation of the ranks of arbitrary wreath products $\wr_{i=1}^{\infty} A_i$ of finite abelian groups, and the construction of finite topological generating sets for these wreath products (**[H1], [H2], [H4]**),
- introducing the notion of a homogeneous automorphism and its crack, and construction (on basis of these two types of automorphisms) of a universal topological decomposition of the wreath product $\wr_{i=1}^{\infty} A_i$ of finite abelian groups into two isomorphic abelian free groups; investigating the group generated by the union of these two abelian free groups (**[H3]**),
- obtaining a general condition for amenability of groups generated by an arbitrary set of homogeneous automorphisms and their cracks (**[H8]**),
- characterization of those sequences $(G_i, X_i)_{i\geq 1}$ of transitive permutation groups for which the wreath product $\wr_{i=1}^{\infty} G_i$ is generated by a single automaton over the alphabet $X = (X_i)_{i\geq 1}$; obtaining an explicit and universal automaton realization for every such a sequence and finding (by using this realization) the upper bound for the rank of the group $\wr_{i=1}^{\infty} G_i$ and for the number of states in the corresponding optimal automaton (**[H7]**),
- characterization of those sequences $(G_i)_{i\geq 1}$ of transitive permutation groups on a finite set X for which the wreath product $\wr_{i=1}^{\infty} G_i$ is generated by a Mealy automaton over X (**[H7]**).

The constructions from the papers [H2] and [H6] are universal enough to use them for arbitrary sequences $(H_i, X_i)_{i\geq 1}$ of non-abelian transitive simple groups and for arbitrary sequences $(A_i, X_i)_{i\geq 1}$ of abelian transitive groups for which the wreath product $\wr_{i=1}^{\infty} A_i$ is topologically finitely generated. Besides, in the case of non-abelian groups, the constructed set is always minimal, and in the case of abelian groups the construction from the paper [H2] is minimal provided that the group A_1 is cyclic.

2.1 Automata for wreath powers of perfect groups – paper [H5]

The infinite wreath power $\wr_{i=1}^{\infty} H^{(i)}$ of a transitive permutation group (H, X) on a finite set X is topologically finitely generated if and only if the group H is perfect ([15]). If we additionally assume that the action of H on X is not free, then the above described Bondarenko's construction gives a finite set $S = \{r_1, \ldots, r_m, d_1, \ldots, d_m\}$ of topological generators for this wreath power. If $m := d(H)$, then every automorphism r_j has exactly two sections: r_j and id_{X^*}, and every automorphism d_j has three sections: d_j, r_j and id_{X^*}. Thus, it is possible to construct a Mealy automaton $A = (\mathcal{S}, X, \varphi, \psi)$ with the $(2m + 1)$-element set of states

$$\mathcal{S} := \{R_1, \ldots, R_m, D_1, \ldots, D_m, Id\},$$

such that the equalities hold: $\widetilde{R_j} = r_j$, $\widetilde{D_j} = d_j$ for $1 \leq j \leq m$ and $\widetilde{Id} = id_{X^*}$. The transition and output functions in A can be formally defined as follows:

$$\begin{aligned} \varphi(D_j, x) &= \begin{cases} D_j, & x = x_1, \\ R_j, & x = x_2, \\ Id, & x \in X \setminus \{x_1, x_2\}, \end{cases} \qquad \varphi(R_j, x) = \varphi(Id, x) = Id, \\ \psi(Id, x) &= \psi(D_j, x) = x, \quad \psi(R_j, x) = h_j(x) \end{aligned}$$

for all $1 \leq j \leq m$ and $x \in X$. Obviously, the automaton A generates $\wr_{i=1}^{\infty} H^{(i)}$. It is not a minimal automaton, because if H is simple, then $|\mathcal{S}| = 5$ and $d(\wr_{i=1}^{\infty} H^{(i)}) = 2$. It is an open question if there is a group H such that the above construction gives a minimal or optimal automaton for the corresponding wreath power.

We solved in [H5] (sections 3–4) the problem of existence of a minimal automaton for the infinite wreath power of a nontrivial group. We based the positive solution on completely different types of automorphisms. For this purpose, we considered k-transitive ($k \geq 1$) perfect groups (H, X) which can be generated by two permutations such that one of them (denoted by β) has all cycles of the same length in the decomposition into disjoint cycles, the number of these cycles is equal to k, and each of these cycles contains a fixed point of the other generator (denoted by α). Let $X' \subseteq X$ be the set of all these fixed points.

Theorem 5 ([H5], Theorem 3) *The Mealy automaton $A = (\{a, b\}, X, \varphi, \psi)$ defined as follows:*

$$\varphi(s, x) = \begin{cases} a, & s = a,\ x \in X', \\ b, & s = b \text{ otherwise}, \end{cases} \qquad \psi(a, x) = \alpha(x), \quad \psi(b, x) = \alpha \circ \beta(x),$$

generates the wreath power $\wr_{i=1}^{\infty} H^{(i)}$. In particular A is a minimal automaton for this wreath power.

We used the above construction ([H5], Proposition 5) for the alternating groups, as for every $n \geq 5$, we have $Alt(n) = \langle \alpha_n, \beta_n \rangle$, where

$$\alpha_n := (1, 2, 3), \quad \beta_n := \begin{cases} (1, 2, \ldots, n), & 2 \nmid n, \\ (1, 3, \ldots, n-1) \circ (2, 4, \ldots, n), & 2 \mid n. \end{cases}$$

The observation that α_n and β_n generate $Alt(n)$ was derived in [P4] (the description of [P4] is given on p. 42).

By using the same reasoning as in ([H5], Proposition 5), we can extend the above construction to an arbitrary sequence $(Alt(m_i))_{i\geq 1}$ of alternating groups of degree $m_i \geq 5$. Consequently, we obtain a minimal automaton for the subgroup $Aut_e(X^*) \leq Aut(X^*)$ of alternating automorphisms.

Theorem 6 *If $X = (X_i)_{i\geq 1}$ is a changing alphabet and $|X_i| \geq 5$ for every $i \geq 1$, then there is a 2-state automaton A over X which generates the wreath product $\wr_{i=1}^{\infty} Alt(X_i)$. If we denote $X_i := \{1, \ldots, m_i\}$ for $i \geq 1$, then the automaton A can be defined as follows:*

$$A = (\{a, b\}, (X_i)_{i\geq 1}, (\varphi_i)_{i\geq 1}, (\psi_i)_{i\geq 1}),$$

where the transition and output functions are defined as follows:

$$\varphi_i(s, x) = \begin{cases} a, & s = a,\ x = m_i, \\ a, & s = a,\ x = m_i - 1,\ 2 \mid m_i, \\ b, & \text{otherwise}, \end{cases} \qquad \psi_i(a, x) = \alpha_{m_i}(x), \quad \psi_i(b, x) = \alpha_{m_i} \circ \beta_{m_i}(x).$$

2.2 The method of wreath recursions – paper [H6]

To study the groups generated by automata from the papers [H3], [H5] and [H6], we use, among others, the method of wreath recursions. In the case of Mealy automata, this is a popular tool to study the groups generated by such automata. In [H6], we introduced the extension of this method to arbitrary automata (i.e. to automata over a changing alphabet). However, for the first time, we used this idea for the constructions from [D5] and [H3].

In the method of wreath recursions, we consider the i-th transition ($i \geq 1$) of the automaton $A = (S, X, \varphi, \psi)$, that is the automaton

$$A_i := (S, (X_j)_{j\geq i}, (\varphi_j)_{j\geq i}, (\psi)_{j\geq i}).$$

Let us denote by $(s)_i$ the automaton transformation defined by the state $s \in S$ of the automaton A_i. In particular, we have $s_i \in Aut(X^*_{(i)})$ (in the sequel, when considering some concrete examples of automata A, we will simply write s_i instead of $(s)_i$, and if A is a Mealy automaton, then we will identify every its state s with the corresponding automaton transformation). Let us denote

$$S := \{\mathrm{s}_1, \ldots, \mathrm{s}_k\}, \quad X_i := \{x_{i,1}, \ldots, x_{i,m_i}\}, \quad i \geq 1. \tag{3}$$

According to the definition of an automaton transformation, we can built the following infinite sequence $\mathcal{R} = (\mathcal{R}_i)_{i\geq 1}$ of finite systems $\mathcal{R}_i$ of wreath recursion of the generators

$$(\mathrm{s}_1)_i, \ldots, (\mathrm{s}_k)_i \in Aut(X^*_{(i)})$$

of the group $G(A_i)$ generated by the automaton A_i:

$$\mathcal{R}_i\colon \begin{cases} (\mathrm{s}_1)_i &= ((\mathrm{q}_{i,1,1})_{i+1}, (\mathrm{q}_{i,1,2})_{i+1}, \ldots, (\mathrm{q}_{i,1,m_i})_{i+1})\pi_{i,1}, \\ (\mathrm{s}_2)_i &= ((\mathrm{q}_{i,2,1})_{i+1}, (\mathrm{q}_{i,2,2})_{i+1}, \ldots, (\mathrm{q}_{i,2,m_i})_{i+1})\pi_{i,2}, \\ \vdots & \quad \vdots \\ (\mathrm{s}_k)_i &= ((\mathrm{q}_{i,k,1})_{i+1}, (\mathrm{q}_{i,k,2})_{i+1}, \ldots, (\mathrm{q}_{i,k,m_i})_{i+1})\pi_{i,k}, \end{cases} \qquad i \geq 1, \tag{4}$$

where $\mathrm{q}_{i,j,r} := \varphi_i(\mathrm{s}_j, x_{i,r}) \in S$, $\pi_{i,j} := \sigma_{\mathrm{s}_j,i} \in Sym(X_i)$ for $1 \leq j \leq k$, $1 \leq r \leq m_i$. Conversely, for every finite set of symbols S and a changing alphabet $X = (X_i)_{i\geq 1}$, defined as in (3), the sequence $\mathcal{R} = (\mathcal{R}_i)_{i\geq 1}$ of finite systems of equations (4), where $\mathrm{q}_{i,j,r} \in S$, $\pi_{i,j} \in Sym(X_i)$, defines uniquely an automaton A over the alphabet $X = (X_i)_{i\geq 1}$ and with the set S as a set of states. We denote such an automaton as a triple

$$A = (S, X, \mathcal{R}),$$

and refer to the sequence $\mathcal{R} = (\mathcal{R}_i)_{i\geq 1}$ as a system of wreath recursions of the automaton A. If A is a Mealy automaton, then $(A_i)_{i\geq 1}$ and $(\mathcal{R}_i)_{i\geq 1}$ are constant sequences, and we get just one system of equations:

$$\mathcal{R}\colon \begin{cases} \mathrm{s}_1 &= (\mathrm{q}_{1,1}, \mathrm{q}_{1,2}, \ldots, \mathrm{q}_{1,m_1})\pi_1, \\ \mathrm{s}_2 &= (\mathrm{q}_{2,1}, \mathrm{q}_{2,2}, \ldots, \mathrm{q}_{2,m_1})\pi_2, \\ \vdots & \quad \vdots \\ \mathrm{s}_k &= (\mathrm{q}_{k,1}, \mathrm{q}_{k,2}, \ldots, \mathrm{q}_{k,m_1})\pi_k, \end{cases}$$

where $\mathrm{q}_{j,r} := \varphi(\mathrm{s}_j, x_{1,r})$, $\pi_j := \sigma_{\mathrm{s}_j}$, $1 \leq j \leq k$, $1 \leq r \leq m_1$.

For example, a $(2m+1)$-state Mealy automaton for the Bondarenko's construction from the previous section can be described by the following system $\mathcal{R}$ of wreath recursions (we assume that the letters x_1 and x_2 are, respectively, in the first and the second positions in X):

$$\mathcal{R}\colon \begin{cases} Id &= (Id, Id, Id, \ldots, Id), \\ R_1 &= (Id, Id, Id, \ldots, Id)h_1, \\ \vdots & \quad \vdots \\ R_m &= (Id, Id, Id, \ldots, Id)h_m, \\ D_1 &= (D_1, R_1, Id, \ldots, Id), \\ \vdots & \quad \vdots \\ D_m &= (D_m, R_m, Id, \ldots, Id), \end{cases}$$

whereas the minimal automaton A from Theorem 6 can be presented as the automaton

$$A = (\{a, b\}, X, \mathcal{R})$$

in which the system $\mathcal{R} = (\mathcal{R}_i)_{i\geq 1}$ of wreath recursions is of the form

$$\mathcal{R}_i\colon \begin{cases} a_i = (b_{i+1}, \ldots, b_{i+1}, c_{i+1}, a_{i+1})\alpha_{m_i}, \\ b_i = (b_{i+1}, \ldots, b_{i+1}, b_{i+1}, b_{i+1})\alpha_{m_i} \circ \beta_{m_i}, \end{cases} \qquad i \geq 1,$$

where $c_{i+1} := \begin{cases} a_{i+1}, & 2 \mid m_i, \\ b_{i+1}, & 2 \nmid m_i. \end{cases}$

In the method of wreath recursions, we study the relations between the elements of the group $G(A_i)$ $(i \geq 1)$ and its sections. For this purpose, we consider these elements as the group-words W over the corresponding generating set as an alphabet, and we use the combinatorial methods for these group-words, involving the notion of the section $W_{\{w\}}$ at any word $w \in X^*_{(i)}$. To define this notion, we consider the generating set

$$S_i := \{(\mathrm{s}_1)_i, \ldots, (\mathrm{s}_k)_i\}, \quad i \geq 1$$

of the group $G(A_i)$ as a basis of the free group $F(S_i)$ of rank k. Next, when considering the equalities (4), we see that every group-word $W \in F(S_i)$ is a product of some of their left sides or their inverses. Thus W defines uniquely a product of some of their right sides or their inverses. Computing this product in accordance with the formulae (1)–(2), we obtain the relation

$$W = (V_1, \ldots, V_{m_i})\pi$$

for some group-words $V_r \in F(S_{i+1})$ $(1 \leq r \leq m_i)$ and a permutation $\pi \in Sym(X_i)$. The group-words V_r are defined uniquely by the group-word W and the system $\mathcal{R} = (\mathcal{R}_i)_{i\geq 1}$. Thus, these words are not usually reduced, which means that they may contain trivial subwords, that is the words of the form $(\mathrm{s}_j)^{\eta}_{i+1}(\mathrm{s}_j)^{-\eta}_{i+1}$ $(\eta \in \{-1, 1\},\ 1 \leq j \leq k)$, as well as the words $(\mathrm{s}_j)^{\eta}_{i+1}$ defining the identity $id_{X^*_{(i+1)}}$ (we do not exclude the trivial states in the automaton A_{i+1}). By consecutive deleting in the word V_r $(1 \leq r \leq m_i)$ all these trivial subwords, we get a unique and reduced form of this word, which we denote by $\widehat{V}_r$ and call the section of W at the letter $x_{i,r} \in X_i$ (defined by the system $\mathcal{R}$). We now define the section $W_{\{w\}} \in F(S_{i+|w|})$ of a group-word $W \in F(S_i)$ at any word $w \in X^*_{(i)}$ recursively: $W_{\{\epsilon\}} := \widehat{W}$ and $W_{\{vx\}} := (W_{\{v\}})_{\{x\}}$ for all $v \in X^*_{(i)}$ and $x \in X_{i+|v|}$.

If a group-word $W \in F(S_i)$ defines an automorphism $g \in Aut(X^*_{(i)})$, then for every $w \in X^*_{(i)}$ the section $W_{\{w\}} \in F(S_{i+|w|})$ defines the automorphism $g_{\{w\}} \in Aut(X^*_{(i+|w|)})$. We also have

$$(WV)_{\{w\}} = \widehat{W_{\{w\}}V_{\{w'\}}},\ (W^{-1})_{\{w\}} = (W_{\{w''\}})^{-1}, \tag{5}$$

for all $W, V \in F(S_i)$ and $w \in X^*_{(i)}$, where $w' := W(w)$, $w'' := W^{-1}(w)$. Moreover, the relation $v \prec w$ implies $|W_{\{w\}}| \leq |W_{\{v\}}|$.

Definition 11 We call a group-word $W \in F(S_1)$ defining the neutral element in $G(A)$ a disappearing group-word, if there is $n \geq 0$ such that $W_{\{w\}} = \epsilon$ for every $w \in X^n$. We refer to the smallest n with this property as the depth of W and denote it by $\lambda(W)$.

By (5), we get that the depth $\lambda\colon F(S_1) \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ satisfies:

$$\lambda(W) = \lambda(\widehat{W}) = \lambda(W^{-1}) = \lambda(U^{-1}WU), \quad \lambda(WV) \leq \max\{\lambda(W), \lambda(V)\} \tag{6}$$

for all $U, V, W \in F(S_1)$. Obviously, not every group-word defining the neutral element in $G(A)$ must be disappearing. By the relations (6), we obtain:

Proposition 1 ([H6], Proposition 3) *If the group $G(A)$ generated by an automaton $A = (S, X, \mathcal{R})$ satisfies the following two conditions:*

(i) every group-word in $F(S_1)$ defining the neutral element is disappearing,

(ii) there are disappearing group-words which have arbitrarily large depth,

then $G(A)$ is not finitely presented.

Indeed, suppose, contrary, that there exists a finite set $R \subseteq F(S_1)$ of defining relations for $G(A)$. By (i), for any nonempty and reduced group-word $W \in F(S_1)$ defining an identity, there are $n \geq 1$ and $V_i \in F(S_1)$, $U_i \in R \cup R^{-1}$ $(1 \leq i \leq n)$ such that W is freely equivalent to $V_1U_1V_1^{-1} \cdot \ldots \cdot V_nU_nV_n^{-1}$. But then, from (6), we get: $\lambda(W) \leq \max_{1\leq i\leq n} U_i \leq L$, where $L := \max_{U\in R} U < \infty$, which contradicts (ii).

The groups generated by automata over a fixed alphabet and satisfying (i) belong to the larger class of groups (piecewise automatically presented groups) investigated by A. Erschler ([24]). She proved that if such a group has the Kazhdan (T) property, then it must be finite, and if it is an amenable and finitely presented group, then it must be virtually abelian (i.e. contains abelian subgroup of finite index).

In [H6], we introduced the notion of an automaton with a path-active nucleus (here, for simplicity, we will call it an automaton with a nucleus). To this aim, for any subset $T \subseteq F(S_1)$, we denote

$$T_{(i)} := \{W_{(i)} \colon W \in T\} \subseteq F(S_i), \quad i \geq 1,$$

where $W_{(i)} \in F(S_i)$ is obtained from the word $W \in F(S_1)$ by replacing every letter $(\mathrm{s}_j)_1^\eta$ with $(\mathrm{s}_j)_i^\eta$ ($1 \leq j \leq k$, $\eta \in \{-1, 1\}$).

Definition 12 An automaton $A = (S, X, \mathcal{R})$ is called an automaton with a nucleus if there is a finite set $T \subseteq F(S_1)$ (called a nucleus of the automaton) of nonempty and reduced group-words which satisfies the following conditions:

(i) for every $W \in F(S_1)$ there is $n \geq 0$ such that $W_{\{w\}} \in T_{(n+1)} \cup \{\epsilon\}$ for all $w \in X^n$,

(ii) for every $W \in T$ there is $u = x_1x_2 \ldots \in X^\omega$ such that for every $i \geq 1$, we have: $W_{(i)}(x_i) \neq x_i$, $(W_{(i)})_{\{x_i\}} = W_{(i+1)}$ and $(W_{(i)})_{\{x\}} = \epsilon$ for $x \in X_i \setminus \{x_i\}$.

Proposition 2 ([H6], Proposition 5) *In a group generated by an automaton $A = (S, X, \mathcal{R})$ with a nucleus every group-word defining the neutral element is disappearing.*

The notion of a nucleus for time-varying automata is analogous to the notion of a contraction in the class of self-similar groups. In particular, every group generated by a Mealy automaton with a nucleus must be contracting.

In the paper [H6], we also extended to time-varying automata the concepts of self-replication and branching, which are well-known in the class of self-similar branch groups ([60]). We based them on the observation that the equalities (4) define for every $i \geq 1$ an embedding

$$\Psi_{i,A} \colon G(A_i) \hookrightarrow G(A_{i+1}) \wr_{X_i} Sym(X_i),$$

which, in general, is not an isomorphism. Let us consider the restriction of the embedding $\Psi_{i,A}$ to the stabilizer of the first level, together with the projection $\Psi_{i,r,A} \colon Stab_{G(A_i)}(1) \to G(A_{i+1})$ on the r-th ($1 \leq r \leq m_i$) coordinate.

Definition 13 If $\Psi_{i,r,A}(Stab_{G(A_i)}(1)) = G(A_{i+1})$ for all $i \geq 1$ and $1 \leq r \leq m_i$, then the automaton A is called self-replicating. If there is a sequence $(K_i)_{i \geq 1}$ of finite index normal subgroups $K_i \lhd G(A_i)$ satisfying $K_{i+1}^{m_i} \leq \Psi_{i,A}(K_i)$ for every $i \geq 1$, then the automaton A is called regularly branch. If the groups K_i are only nontrivial, then A is called regularly weakly branch.

In particular, if the automaton A is regularly branch and the group $G(A)$ acts spherically transitively, then this is a branch group (in a sense of Definition 6), and if A is regularly weakly branch, then $G(A)$ is a weakly branch group. If additionally A is a Mealy-type automaton, then in the first case the group $G(A)$ is regularly branch, and in the second case it is regularly weakly branch (in a sense of Definition 7).

In [H6], we also introduced the notion of a nearly finitary automorphism and a nearly finitary group.

Definition 14 An automorphism $g \in Aut(X^*)$ is called nearly finitary if there is $n := n_g \geq 0$ and a finite set $T := T_g \subseteq X^\omega$ (perhaps empty) such that $T \cap Fix(g) = \emptyset$, and the vertex permutations $\sigma_{g,w}$ are trivial for all words $w \in X^*$ of length greater than n which are not a prefix of any word in T. If every element of a group $G \leq Aut(X^*)$ is a nearly finitary automorphism, then we call G a nearly finitary group.

Proposition 3 ([H6], Proposition 4) *If $A = (S, X, \mathcal{R})$ is an automaton with a nucleus, then $G(A)$ is a nearly finitary group.*

The set of all nearly finitary automorphisms of the tree X^* contains properly the group of finitary automorphisms, which in turn forms a dense subgroup of the group $Aut(X^*)$ (an automorphism $g \in Aut(X^*)$ is called finitary if the vertex permutations $\sigma_{g,w}$ are nontrivial for at most finitely many vertices $w \in X^*$). However, the set of nearly finitary automorphisms does not form a group (admittedly the inverse of a nearly finitary automorphism is a nearly finitary automorphism but the composition of two such automorphisms need not be of this type). On the other hand, the set of nearly finitary automorphisms is properly contained in the group of bounded automorphisms of the tree X^*.

Definition 15 An automorphism $g \in Aut(X^*)$ is called bounded if the sequence $(\xi_g(n))_{n\geq 1}$ is bounded, where $\xi_g(n) := |\{w \in X^n \colon g_{\{w\}} \neq id_{X^*_{(n+1)}}\}|$.

In 2010, V. Nekrashevych ([61]) showed that if the alphabet $X = (X_i)_{i\geq 1}$ is bounded, then the group of all bounded automorphisms does not contain non-abelian free subgroups (in the case of an unbounded alphabet X, it is not difficult to construct a non-abelian free group in the group of bounded automorphisms of the tree X^*). The proof was based on the following alternative:

Theorem 7 (Nekrashevych, [61]) *Let G be a group acting faithfully on an infinite locally finite rooted tree T. Then one of the following holds:*

(1) G does not contain non-abelian free subgroups,

(2) there is a non-abelian free group $F \leq G$ and an infinite path u starting in the root of T such that $F \subseteq Stab_G(u)$ and F acts faithfully on every neighborhood of u (i.e. on a subtree of T which consists of all descendants of some prefix of u),

(3) there is an infinite path u starting in the root of T and a non-abelian free subgroup $F \leq G$ such that the stabilizer $Stab_F(u)$ is trivial.

We used the above alternative to show the following theorem.

Proposition 4 ([H6], Proposition 1) *If $X = (X_i)_{i\geq 1}$ is an arbitrary changing alphabet (bounded or unbounded) and $G \leq Aut(X^*)$ is a nearly finitary group, then G does not contain non-abelian free subgroups.*

Remark 4 The method of wreath recursions, although quite useful, might be insufficient even towards very simple automata. For example, if we take the group generated by a 2-state Mealy automaton $A = (\{a, b\}, \{1, 2, 3\}, \mathcal{R})$ with the following system of wreath recursions

$$\mathcal{R}\colon \begin{cases} a = (a, b, a)(1, 2, 3), \\ b = (a, a, b)(1, 2), \end{cases}$$

then we even do not know whether $G(A)$ is finite (it is supposed that this is an infinite group). The above automaton (denoted by $\mathcal{M}_{675}$) appears in the classification of groups generated by a 2-state Mealy automaton over a 3-letter alphabet ([59]).

2.3 The automaton $\mathcal{A}$ and the group $G(\mathcal{A})$ – cont. of [H6]

Let $X = (X_i)_{i\geq 1}$ be a changing alphabet. The main result of the paper [H6] is the universal construction of an almost minimal automaton for the wreath product $\wr_{i=1}^{\infty} H_i$, where $(H_i, X_i)_{i\geq 1}$ is an arbitrary sequence of non-abelian simple transitive permutation groups. The construction is based on the observation that if (H, Y) is a non-abelian simple transitive permutation group on a finite set Y, then there is a triple $(\alpha, \beta, y) \in H \times H \times Y$ (which we called a hooked triple) satisfying the following conditions: $H = \langle \alpha, \beta \rangle$, $o(\alpha) = 2$, $o(\beta) = p$ for some prime number $p \geq 3$, and in the decompositions of the permutations α and β into the products of disjoint cycles, there are two nontrivial cycles, one in the decomposition of α and the second in the decomposition of β, such that y is the only common element in both these cycles. First of all, note that the existence of such a triple (α, β, y) follows directly from the existence of the elements $\alpha, \beta \in H$ satisfying the following two conditions:

(i) $H = \langle \alpha, \beta \rangle$, $o(\alpha) = 2$, $o(\beta) = p$ for some prime $p \geq 3$,

(ii) $p < |Y|$.

Indeed, let us take the permutations $\alpha, \beta \in H$ satisfying (i)–(ii), and consider their decompositions $\alpha = \tau_1 \cdot \ldots \cdot \tau_s$, $\beta = \sigma_1 \cdot \ldots \cdot \sigma_t$ into the products of nontrivial disjoint cycles. Then there is a transposition τ_{j_0} which has exactly one common element (denote it by y) with a cycle σ_{k_0} for some $1 \leq k_0 \leq t$ (otherwise the support of every τ_j either would be contained in $Fix(\beta)$ or it would be contained in the support of some $\sigma_{j'}$, but then, by (ii), the group $H = \langle \alpha, \beta \rangle$ would not be transitive). Consequently (α, β, y) is a hooked triple. The observation that there are elements $\alpha, \beta \in H$ satisfying (i) is the main result due to C. S. H. King ([49]). Let us show that the elements α, β can be chosen in such a way that also the condition (ii) is satisfied. So, let us choose $\alpha, \beta \in H$ satisfying (i) and let us assume that the prime number $p = o(\beta) \geq 3$ is possible minimal. We obviously have $|Y| \geq 5$ and $|Y| \geq p$. If $2 \mid |Y|$, then $p < |Y|$. Thus, we can assume that $2 \nmid |Y|$. Suppose contrary that $p = |Y|$. Then β is a long cycle (i.e. a cycle of length $|Y|$). We use the following lemma:

Lemma 1 ([H6], Lemma 1) *Every finite simple permutation group with a long cycle is a primitive group.*

There is a classification of primitive permutation groups containing a long cycle (below, we use the standard notation for the general affine group $AGL_k(q)$ of degree k over a q-element field, the projective general linear group $PGL_k(q)$, the projective semilinear group $P\Gamma L_k(q)$, the projective special linear group $PSL_k(q)$, and the Mathieu groups M_{11} and M_{23}):

Theorem 8 ([44]) *If G is a primitive group of degree m containing a cycle of length m, then one of the following holds:*

- *$G \leq AGL_1(m)$, where m is a prime number,*
- *$G = Alt(m)$ or $G = Sym(m)$,*

- $PGL_k(q) \leq G \leq P\Gamma L_k(q)$, *where* $k \geq 2$, q *is a prime number and* $m = (q^k-1)/(q-1)$,
- $m = 11$ *and* $G = PSL_2(11)$ *or* $G = M_{11}$,
- $m = 23$, $G = M_{23}$.

In above, if $m := p$ is a prime number, then $AGL_1(m)$ contains a cyclic group of order m, which is a normal subgroup. Besides of that the group $PGL_k(q)$ is a normal subgroup of $P\Gamma L_k(q)$ and $PGL_k(q)$ is simple if and only if $GCD(k, q-1) = 1$ and $(k, q) \neq (2, 2)$, which implies $PGL_k(q) = PSL_k(q)$. Since H is non-abelian and simple, we obtain the following possibilities:

- $H = Alt(p)$, or
- $p = 23$ and $H = M_{23}$, or
- $p = 11$ and $H = M_{11}$, or
- $p = 11$ and $H = PSL_2(11)$, or
- $p = (q^k-1)/(q-1)$ and $H = PSL_k(q)$, where $k \geq 2$ and q is a prime power satisfying $GCD(k, q-1) = 1$.

Now, it is enough to observe that the group $PSL_2(11)$ and the groups $Alt(p)$ for $p \neq 7$, as well as the groups $PSL_k(q)$ with k and q as above are generated by an involution and an element of order 3; and the groups $Alt(7)$, M_{11} and M_{23} are generated by an involution and an element of order 5 ([49, 66]). In every case, we obtain a contradiction with the assumption that p is minimal.

Fore every $i \geq 1$ let us take an arbitrary hooked triple

$$(\alpha_i, \beta_i, x_i) \in H_i \times H_i \times X_i,$$

and let us consider a 3-state automaton $\mathcal{A} = (\{a, b, id\}, (X_i)_{i\geq 1}, (\mathcal{R}_i)_{i\geq 1})$ with the following system $\mathcal{R} = (\mathcal{R}_i)_{i\geq 1}$ of wreath recursions:

$$\mathcal{R}_i\colon \begin{cases} a_i &= (a_{i+1}, id_{i+1}, \ldots, id_{i+1})\alpha_i, \\ b_i &= (b_{i+1}, id_{i+1}, \ldots, id_{i+1})\beta_i, \qquad i \geq 1, \\ id_i &= (id_{i+1}, id_{i+1}, \ldots, id_{i+1}), \end{cases} \tag{7}$$

where above, we assume the orderings of the alphabets X_i ($i \geq 1$) in such a way that the letter $x_i \in X_i$ from the triple (α_i, β_i, x_i) occurs in the first position.

First of all, observe that in spite of the fact that the nontrivial vertex permutations of the automorphisms $\widetilde{a}$ and $\widetilde{b}$ are placed along the infinite word $x_1x_2x_3\ldots \in X^\omega$, these automorphism are not directed, as they do not fix this word.

Theorem 9 ([H6], Theorem 1) *The transformations* $\widetilde{a}, \widetilde{b} \in Aut(X^*)$ *defined by the states* a *and* b *of the automaton* $\mathcal{A}$ *topologically generate the wreath product* $\wr_{i=1}^{\infty} H_i$*. In particular* $\mathcal{A}$ *is an almost minimal automaton for the wreath product* $\wr_{i=1}^{\infty} H_i$.

For the proof, we consider the i-th transition $\mathcal{A}_i = (\{a, b, id\}, (X_l)_{l\geq i}, (\mathcal{R}_l)_{l\geq i})$ of the automaton $\mathcal{A}$. We obviously have $G(\mathcal{A}_i) = \langle a_i, b_i\rangle \leq W_i$ for every $i \geq 1$, where $W_i := \wr_{l=i}^{\infty} H_l$. For $i, j \geq 1$ let $W_{i,j} \leq W_i$ be the subgroup obtained from W_i by replacing the vertex permutations $\sigma_{g,w}$ ($g \in W_i$) at the vertices $w \in X_{(i)}^*$ with $|w| \geq j$ by the trivial permutations. Let

$a_{i,j}, b_{i,j} \in W_{i,j}$ be elements obtained in this way from the generators $a_i, b_i \in W_i$. We need to show the equality $\langle a_{i,j}, b_{i,j}\rangle = W_{i,j}$ for all $i, j \geq 1$. We use induction on j. For $j = 1$, we have: $W_{i,1} = H_i = \langle \alpha_i, \beta_i\rangle = \langle a_{i,1}, b_{i,1}\rangle$ for all $i \geq 1$. Let us assume that there is $j_0 \geq 1$ such that $W_{i,j_0} = \langle a_{i,j_0}, b_{i,j_0}\rangle$ for every $i \geq 1$. Let us choose arbitrarily $i_0 \geq 1$, and denote $g := a_{i_0,j_0+1}$, $g' := a_{i_0+1,j_0}$, $h := b_{i_0,j_0+1}$, $h' := b_{i_0+1,j_0}$, $id := id_{i_0+1}$, $\alpha := \alpha_{i_0}$, $\beta := \beta_{i_0}$. By (7), we have:

$$\begin{cases} g = (g', id, \ldots, id)\alpha, \\ h = (h', id, \ldots, id)\beta. \end{cases} \tag{8}$$

Thus, we need to show that the group W_{i_0,j_0+1} is contained in the group $G := \langle g, h\rangle$ (the converse inclusion is obvious). We obviously have $W_{i_0,j_0+1} = W_{i_0+1,j_0} \wr_{X_{i_0}} H_{i_0} = W_{i_0+1,j_0}^{X_{i_0}} \rtimes H_{i_0}$, and, by the inductive assumption, we get $W_{i_0+1,j_0} = \langle g', h'\rangle$. Since W_{i_0+1,j_0} is perfect, for every $f \in W_{i_0+1,j_0}$, there is a group-word $V(x, y) \in F(x, y)$ in which vanish the sums of exponents on the letters x and y, as well as the equality $f = V(g', h')$ is satisfied. Let us compute the wreath recursion of the element $V(g^2, h^{p_{i_0}}) \in G$. By wreath recursions (8) and the construction of the hooked triple (α, β, x_{i_0}), we get $V(g^2, h^{p_{i_0}}) = (f, id, \ldots, id)$. Thus $(f, id, \ldots, id) \in G$. Since f was chosen arbitrarily, we also get: $(g', id, \ldots, id) \in G$ and $(h', id, \ldots, id) \in G$. Consequently, by (8), we get: $\alpha, \beta \in G$, and hence $H_{i_0} \leq G$. Since the group H_{i_0} is transitive, we also get $W_{i_0+1,j_0}^{X_{i_0}} \leq G$. Hence, we obtain $W_{i_0,j_0+1} \leq G$, and the inductive argument finishes the proof.

In [H6], we also derived the following result:

Proposition 5 ([H6], Propositions 8, 10) *The automaton $\mathcal{A}$ is an automaton with a 6-element nucleus*

$$T = \{a_1, a_1^{-1}, b_1, b_1^{-1}, a_1^{-1}b_1, b_1^{-1}a_1\}.$$

In particular, every group-word $W \in F(\{a_1, b_1, id_1\})$ defining the neutral element in the group $G(\mathcal{A})$ is disappearing. Moreover, there are disappearing group words $W \in F(\{a_1, b_1, id_1\})$ with an arbitrary large depth.

For the proof of the second part of Proposition 5, we construct for every $i \geq 1$ the group-words on the letters a_i and b_i of the form:

$$W_{M,N,i} := [b_i^N a_i^{-M} b_i^{-N}, a_i^N] := b_i^N a_i^M b_i^{-N} a_i^{-M} b_i^N a_i^{-M} b_i^{-N} a_i^M,$$

where the exponents M and N satisfy: $2 \mid M$ and $p_i \mid N$, where $p_i := o(\beta_i)$. Next, we show that $W_{M,N,i}(x) = x$ for every $x \in X_i$, and compute the corresponding one-letter sections:

$$(W_{M,N,i})_{\{x\}} = \begin{cases} W_{M/2,N/p_i,i+1}, & x = x_i, \\ \varepsilon, & x \neq x_i. \end{cases}$$

By above, we obtain that for every $i \geq 1$ the group-word $W_{2^{i+1},p_1p_2\ldots p_i,1}$ is disappearing and its depth is equal to $i + 1$. As a result of Propositions 1–5, we obtain:

Theorem 10 ([H6], Theorems 2, 3) *$G(\mathcal{A})$ is a nearly finitary not finitely presented group without non-abelian free subgroups.*

Proposition 5 can be also used to show that the semigroup generated by the maps $\widetilde{a}$ and $\widetilde{b}$ is free ([H6], Proposition 9). Indeed, let $W \in F(\{a_i, b_i\})$ $(i \geq 1)$ be a reduced semigroup-word (i.e. without negative exponents). By induction on the length of W, we show on basis of the wreath recursions (7) that if W begins with a_i (resp. with b_i), then the section $W_{\{x_i\}}$ (which is a reduced semigroup-word in the letters a_{i+1} and b_{i+1}) begins with a_{i+1} (resp. with b_{i+1}). Since the only semigroup-words in the nucleus T of the automaton $\mathcal{A}$ are a_1 and b_1, we conclude that if $W \in F(\{a_1, b_1\})$ is a reduced semigroup-word which begins with a_1 (resp. with b_1), then there is $i \geq 1$ such that $W_{\{x_1 \ldots x_i\}} = a_{i+1}$ (resp. $W_{\{x_1 \ldots x_i\}} = b_{i+1}$). Since $a_i \neq b_i$ for every $i \geq 1$, we conclude that arbitrary two distinct and reduced semigroup-words in the letters a_1 and b_1 can not define the same transformation (we may assume that one of these semigroup-words begins with a_1 and the second with b_1, or one is empty and the second is nonempty), thus the semigroup $sgp(\widetilde{a}, \widetilde{b})$ is free. As a corollary, we obtain the following result:

Corollary 1 ([H6], Corollary 4) *The group $G(\mathcal{A})$ generated by the automaton $\mathcal{A}$ has an exponential growth.*

We have also shown that the automaton $\mathcal{A}$ is self-replicating and regularly weakly branch over the sequence $(G(\mathcal{A}_i)')_{i\geq 1}$ of commutator subgroups.

Proposition 6 ([H6], Proposition 12) *The automaton $\mathcal{A}$ is self-replicating and regularly weakly branch over the sequence $(G(\mathcal{A}_i)')_{i\geq 1}$ of commutator subgroups. In particular, the group $G(\mathcal{A})$ is weakly branch.*

If the sequence $(H_i, X_i)_{i\geq 1}$ is constant, then $\mathcal{A}$ is a Mealy automaton. In addition, this is a bounded automaton, that is every its state defines a bounded automorphism. The concept of boundedness for Mealy automata was studied by S. Sidki ([71]) in 2000. The groups generated by a bounded Mealy automaton form a subclass in the class of self-similar contracting groups. Although the problem of amenability for groups from this wider class is still open, it was proved ([7]) that groups generated by a bounded Mealy automaton are amenable. Consequently, we obtain the following corollary:

Corollary 2 ([H6], Corollary 2) *For the infinite wreath power $\wr_{i=1}^{\infty} H^{(i)}$ of an arbitrary non-abelian simple transitive group H on a finite set X there is an almost minimal automaton realization by a 3-state bounded Mealy automaton $\mathcal{A}$ over X. The group $G(\mathcal{A})$ generated by this automaton is an amenable not finitely presented nearly finitary group of exponential growth, and its two generators corresponding to nontrivial states of $\mathcal{A}$ generate a free semigroup. Moreover, the group $G(\mathcal{A})$ is self-replicating contracting and regularly weakly branch over its commutator subgroup.*

2.4 Generation of wreath products of abelian groups – papers [H1, H2, H4]

In my PhD Thesis, I showed (the paper [D5]) that the wreath products $\wr_{i=1}^{\infty} C_{n_i}$ of finite transitive cyclic groups C_{n_i} with a pairwise coprime orders, that is satisfying the condition $GCD(n_i, n_j) = 1$ for $i \neq j$, are topologically 2-generated. The proof was based on the construction of a minimal automaton (i.e. a 2-state automaton) for such a wreath product. After defending, I wondered about extending this result to wreath products $\wr_{i=1}^{\infty} C_{n_i}$ of arbitrary finite transitive cyclic groups, and I wanted to find the formula for the rank of such a wreath product. I still do not know if there exists the construction of a minimal automaton, but, in the paper [H1], I discovered the formula for the rank of an arbitrary wreath product of this form, as well as I constructed the corresponding minimal topological generating set (the formula is partly a consequence of this construction).

Theorem 11 ([H1], Theorem 1.1) *Let $(n_i)_{i\geq 1}$ be an arbitrary sequence of positive integers greater than* 1. *Then* $d(\wr_{i=1}^{\infty} C_{n_i}) = d(\prod_{i\geq 2} C_{n_i}) + 1$.

Note that the right side of the above equality is equal to the smallest integer k_0 such that the greatest common divisor of an arbitrary k_0-element subsequence of the sequence $(n_i)_{i\geq 2}$ is one. Thus Theorem 11 can be used to determine the ranks of finitely iterated wreath products of the form $\wr_{i=1}^{m} C_{n_i}$ $(m \geq 1)$. In particular, we obtain the estimation $d(\wr_{i=1}^{m} C_{n_i}) \leq m$, and the equality holds if and only if $GCD(n_2, \ldots, n_m) > 1$. In addition, for every $m \geq 2$ there is an infinite sequence $(n_i)_{i\geq 1}$ such that $d(\wr_{i=1}^{\infty} C_{n_i}) = m$. For example, as n_1 we can take an arbitrary integer greater than 1 and as n_i $(i \geq 2)$ we can take the product $p_i p_{i+1} \ldots p_{i+m-2}$, where $(p_i)_{i\geq 2}$ is an arbitrary sequence of pairwise different primes.

Let us denote $\rho := d(\prod_{i\geq 2} C_{n_i})$ and assume $\rho < \infty$ (if $\rho = \infty$, then obviously $d(\wr_{i=1}^{\infty} C_{n_i}) = \infty$). The proof of Theorem 11 is based on the construction of a $(\rho + 1)$-element set

$$S := \{\beta, \alpha_1, \ldots, \alpha_\rho\}$$

of topological generators for the wreath product $\wr_{i=1}^{\infty} C_{n_i}$. For this construction, we consider $\wr_{i=1}^{\infty} C_{n_i}$ as an automorphism group of the tree X^* over a changing alphabet $X = (X_i)_{i\geq 1}$ such that $|X_i| = n_i$ for every $i \geq 1$. We define β as a rooted automorphism generating the cyclic group C_{n_1}. The automorphisms α_j $(1 \leq j \leq k)$ are defined as follows: for the groups C_{n_i} $(i \geq 2)$, we choose ρ-element generating sets $\{a_{1,i}, \ldots, a_{\rho,i}\} \subseteq C_{n_i}$ satisfying the condition: $GCD(o(a_{j,i}), o(a_{j,i'})) = 1$ for all $1 \leq j \leq \rho$ and $i \neq i'$. The existence of these sets, and their construction, were derived in ([H1], Lemma 3.1). Next, for every $i \geq 1$, we choose two letters $x_i, y_i \in X_i$, and define α_j, by its vertex permutations, as follows: $\sigma_{\alpha_j, w_i} := a_{j,i+1}$ for $i \geq 1$ and $\sigma_{\alpha_j, w} := id_{X_{|w|+1}}$ for $w \notin \{w_1, w_2, \ldots\}$, where $w_i := x_1 \ldots x_{i-1} y_i$. In particular, each automorphism α_j is 1-directed and the word $x_1 x_2 x_3 \ldots \in X^\omega$ is its direction.

We divide the main burden of the proof of Theorem 11 into several lemmas ([H1], Lemma 3.2–3.5), which show that the set S indeed topologically generates the wreath product $\wr_{i=1}^{\infty} C_{n_i}$. To this aim, we show at first that for every $n \geq 0$ the group $\langle S\rangle$ acts transitively on the level n of the tree X^* (induction on n). Fix $N \geq 0$. Then by the assumption $GCD(o(a_{j,i}), o(a_{j,i'})) = 1$ for $1 \leq j \leq \rho$, $i \neq i'$, we can use the Chinese remainder theorem to construct for any $1 \leq i \leq N$ and $\gamma \in C_{n_i}$ an element $g := g(i, \gamma) \in \langle S\rangle$ which satisfies: $\sigma_{g, w_{i-1}} = \gamma$ and $\sigma_{g,w} = id_{X_{|w|+1}}$ for $w \in X^{\leq N} \setminus \{w_{i-1}\}$ (we denote by $X^{\leq N}$ the set of words of length not greater than N, we also define $w_0 := \epsilon$). Next, for every $w \in X^{\leq N}$ we consider the subtree $V_w \subseteq X^{\leq N}$ of all words having w as a prefix. By using the construction of the elements $g(i, \gamma)$, as well as the fact that $\langle S\rangle$ acts spherically transitively, we construct for every $w \in X^{\leq N}$ and $\gamma \in C_{n_{|w|+1}}$ an element $h := h(w, \gamma) \in \langle S\rangle$ such that $\sigma_{h,w} = \gamma$ and $\sigma_{h,v} = id_{X_{|v|+1}}$ for all $v \in X^{\leq N} \setminus V_w$. Next, we fix an arbitrary word $w \in X^{\leq N}$ and for any $v \in V_w$, we choose arbitrarily $\gamma_v \in C_{n_{|v|+1}}$. On basis of the elements $h(v, \gamma_v)$, we construct the element $f_w \in \langle S\rangle$ such that $\sigma_{f_w, v} = \gamma_v$ for $v \in V_w$ and $\sigma_{f_w, v} = id_{X_{|v|+1}}$ for $v \in X^{\leq N} \setminus V_w$. For the construction of f_w, we use the backward induction on $|w|$. Namely, if $|w| = N$, then $V_w = \{w\}$ and we define $f_w := h(w, \gamma_w)$. Next, we inductively assume that for some $1 \leq i \leq N$ the required element f_w can be constructed for every word w satisfying $i \leq |w| \leq N$. Finally, on basis of the constructed elements, we construct f_w for every $w \in X^{i-1}$. If we now take $w := \epsilon$, then we have $V_w = V_\epsilon = X^{\leq N}$. Thus, sine N was chosen arbitrarily, we obtain that S topologically generates the wreath product $\wr_{i=1}^{\infty} C_{n_i}$.

For the proof that S is a minimal set, i.e. $|S| = d(\wr_{i=1}^{\infty} C_{n_i})$, we use the Lucchini's formulae obtained in 1997 from the algebraical method due to W. Gaschütz relying on the knowledge of the structure of irreducible G-modules for a finite solvable group G.

Theorem 12 (Lucchini, [55]) *If (G, X) is a transitive permutation group on a finite set X nad H is a finite solvable grup, then*

$$d(H \wr_X G) = \max\left(d(H/H' \wr_X G), \left\lceil \frac{d(H)-2}{|X|} \right\rceil + 2\right).$$

Theorem 13 (Lucchini, [55]) *If (G, X) is a nilpotent transitive permutation group on a finite set X and A is a finite abelian group, then*

$$d(A \wr_X G) = \max_p \{d(A \times G), d(A) + 1, d_p(A) + 2\},$$

where p runs over the set of primes dividing the order of A for which the group G is not p-solvable, and $d_p(A)$ denotes the rank of a Sylow p-subgroup of the group A.

To show that S is minimal, it is enough to show the inequality $d(\wr_{i=1}^{\infty} C_{n_i}) \geq \rho + 1$. For this purpose, we observe that there is $i_0 \geq 1$ such that $d(A) = \rho = d(\prod_{i=2}^{\infty} C_{n_i})$, where $A := \prod_{i=2}^{i_0} C_{n_i}$. Moreover, the group $H := \wr_{i=2}^{i_0} C_{n_i}$ is solvable and $H/H' \simeq A$. Thus, for the wreath product $W := H \wr_{X_1} C_{n_1}$ (which is a homomorphic image of $\wr_{i=1}^{\infty} C_{n_i}$), we get by Theorems 12–13:

$$d(\wr_{i=1}^{\infty} C_{n_i}) \geq d(W) \geq d(H/H' \wr_{X_1} C_{n_1}) = d(A \wr_{X_1} C_{n_1}) \geq d(A) + 1 = \rho + 1.$$

In [H4], we investigated the set $S = \{\alpha_1, \ldots, \alpha_\rho, \beta\}$ as a set of automaton transformations. We have observed that for any non-zero level of the tree X^* the sections of the automorphism β at the words in this level are all trivial, and every transformation α_j $(1 \leq j \leq \rho)$ has at most three distinct sections, and one of them is also trivial. Thus β can be defined by using only one state, and for each automorphism α_j we need only two more states – we do not need new states for the trivial sections of α_j, as we can *tie* every such a section together with the corresponding trivial section of the automorphism β. Consequently, we obtain a $(2\rho + 1)$-state automaton, which generates the wreath product $\wr_{i=1}^{\infty} C_{n_i}$. Admittedly, this construction does not provide a minimal automaton (and probably nor even an optimal automaton), but it allows to characterize the wreath products $\wr_{i=1}^{\infty} C_{n_i}$ as groups generated by an automaton.

Theorem 14 ([H4], Theorem 1) *Let $X = (X_i)_{i \geq 1}$ be a changing alphabet; denote $n_i := |X_i|$ for $i \geq 1$. The following are equivalent:*

(i) the wreath product $\wr_{i=1}^{\infty} C_{n_i}$ is generated by an automaton over X,

(ii) the wreath product $\wr_{i=1}^{\infty} C_{n_i}$ is topologically finitely generated.

If the sequence $(n_i)_{i \geq 1}$ consists of distinct primes, then $\rho = 1$ and $S = \{\beta, \alpha_1\}$. E. Fink ([25]) derived in this case some algebraic and geometric properties of the group $G = \langle S \rangle$. It turns out that even in this simplest case, we do not know whether or not G contains non-abelian free semigroups.

Theorem 15 (Fink, [25]) *If $(n_i)_{i \geq 1}$ is a sequence of distinct primes, then the group $G = \langle \beta, \alpha_1 \rangle$ has the following properties:*

(i) G is a branch group, but it does not have the congruence subgroup property, and every normal subgroup $K \triangleleft G$ is finitely generated,

(ii) the abelianization G/G' is isomorphic to $C_{n_1} \times C_\infty$; in particular G is not just-infinite,

(iii) G is not solvable, but every proper quotient G/K is a solvable group,

(iv) G does not have the polynomial growth, and if $n_i > c^{(2+i)\cdot 3^{i+1}+1}$ for some constant $c > 1$, then G is of exponential growth, but if $n_i > 36^i$, then G does not contain non-abelian free subgroups.

In the paper [H2], we have extended the ideas and the methods from [H1] to the wreath product $\wr_{i=1}^{\infty} A_i$, where $(A_i, X_i)_{i\geq 1}$ is an arbitrary sequence of abelian transitive permutation groups on finite sets X_i. We have observed that the rank $d(\wr_{i=1}^{\infty} A_i)$ can be computed on basis of the Lucchini's formulae. By induction on $n \geq 1$, we derived from Theorems 12–13 the following formula:

$$d(\wr_{i=1}^{n} A_i) = \max\left\{ d(\prod_{i=1}^{n} A_i), 1 + d(\prod_{i=2}^{n} A_i) \right\}, \quad n \geq 1.$$

Note that all the three sequences $(d(\wr_{i=1}^{n} A_i))_{n\geq 1}$, $(d(\prod_{i=1}^{n} A_i))_{n\geq 1}$ and $(1 + d(\prod_{i=2}^{n} A_i))_{n\geq 1}$ stabilize, or all of them diverge to infinity ([68], Lemma 2.5.3). As a result, we obtain the formula for $d(\wr_{i=1}^{\infty} A_i)$.

Theorem 16 ([H2], Theorem 1.1) *For any sequence $(A_i, X_i)_{i\geq 1}$ of abelian transitive permutation groups on finite sets X_i the following equality holds:*

$$d(\wr_{i=1}^{\infty} A_i) = \max\left\{ d(\prod_{i=1}^{\infty} A_i), 1 + d(\prod_{i=2}^{\infty} A_i) \right\}.$$

Further, we have obtained a more general criterion for the topological generation of wreath products of abelian groups, which we based on rooted and directed automorphisms. To this aim, we introduce the notion of a mutually coprime automorphism.

Definition 16 An automorphism $g \in Aut(X^*)$ is called mutually coprime, if for any two vertices w and v from distinct levels of the tree X^* the vertex permutations $\sigma_{g,w}$ and $\sigma_{g,v}$ are of coprime orders: $GCD(o(\sigma_{g,w}), o(\sigma_{g,v})) = 1$.

Proposition 7 ([H2], Proposition 4.5) *Let $X = (X_i)_{i\geq 1}$ be a changing alphabet and let $R \subseteq Aut(X^*)$ be an arbitrary set of rooted automorphisms and $D \subseteq Aut(X^*)$ – an arbitrary (finite or infinite) set of 1-directed mutually coprime automorphisms. Let us denote $S := R \cup D$ and let $V_{S,i} \leq Sym(X_i)$ $(i \geq 1)$ be the vertex groups of the set S, defined as follows:*

$$V_{S,i} := \langle \sigma_{g,w} \colon g \in S, w \in X^{i-1} \rangle.$$

If all the groups $V_{S,i}$ are abelian and transitive, then the set S topologically generates their wreath product: $\overline{\langle S \rangle} = \wr_{i=1}^{\infty} V_{S,i}$.

Conversely, let $(A_i, X_i)_{\geq 1}$ be an arbitrary sequence of abelian transitive permutation groups on finite sets X_i $(|X_i| \geq 2)$. Denote $\rho := d(\prod_{i=2}^{\infty} A_i)$ and assume $\rho < \infty$.

Lemma 2 ([H2], Lemma 5.1) *For every $i \geq 2$, there is a ρ-element generating set*

$$\{\sigma_{1,i}, \ldots, \sigma_{\rho,i}\}$$

of the group A_i such that $GCD(o(\sigma_{j,i}), o(\sigma_{j,i'})) = 1$ for all $1 \leq j \leq \rho$ and $i \neq i'$.

On basis of the generating sets from the above lemma, we constructed 1-directed automorphisms $d_1, \ldots, d_\rho \in Aut(X^*)$ in such a way that for $1 \le j \le \rho$ the only potentially nontrivial vertex permutation of d_j at the vertex in the i-th level ($i \ge 1$) of the tree X^* is equal to $\sigma_{j,i+1}$. Denote $D := \{d_1, \ldots, d_\rho\}$. We also constructed the set R consisting of $d(A_1)$ rooted automorphisms, which generates the group A_1. In particular, for the vertex groups $V_{S,i}$ of the set

$$S := R \cup D,$$

we have: $V_{S,i} = A_i$ for every $i \ge 1$. Since the automorphisms d_j are mutually coprime, we obtained by Proposition 7 that S topologically generates the wreath product $\wr_{i=1}^{\infty} A_i$. If the group A_1 is cyclic, then by Theorem 16, we obtained that S is minimal, i.e. $|S| = d(\wr_{i=1}^{\infty} A_i)$.

Theorem 17 ([H2], Theorem 1.2) *If the group A_1 is cyclic, then the above constructed set $S = R \cup D$ is a minimal topological generating set of the wreath product $\wr_{i=1}^{\infty} A_i$, that is $|S| = d(\wr_{i=1}^{\infty} A_i)$.*

By analogy to the case of cyclic groups, we can use the set $S = R \cup D$ and obtain an automaton generating the wreath product $\wr_{i=1}^{\infty} A_i$. Obviously, this construction does not provide a minimal automaton.

Corollary 3 *Let us denote $n := d(A_1) + 2\rho$. Then there is an n-state automaton A over the alphabet $X = (X_i)_{i \ge 1}$ which generates the wreath product $\wr_{i=1}^{\infty} A_i$.*

2.5 Topological decomposition into abelian free groups – paper [H3]

In the paper [H4] (described in the previous section), we have constructed (on basis of rooted and directed automorphisms) the topological decompositions of wreath products $\wr_{i=1}^{\infty} A_i$ of finite abelian groups into two finitely generated abelian groups, the first being finite (that generated by rooted automorphisms) and equal to A_1, and the second (that generated by 1-directed automorphisms) being infinite (however, from that construction it is not possible to determine the corresponding canonical decomposition into a direct product of cyclic groups).

In general, we say that a group $G \le Aut(X^*)$ decomposes topologically over two its subsets, if the groups generated by these subsets intersect trivially, and the union of these sets generates a dense subgroup of G.

Let $X = (X_i)_{i \ge 1}$ be a changing alphabet and let $(A_i, X_i)_{i \ge 1}$ be an arbitrary sequence of abelian transitive permutation groups such that the rank

$$\rho := d(\prod_{i \ge 1} A_i)$$

is finite (equivalently: $d(\wr_{i=1}^{\infty} A_i) < \infty$). In the paper [H3], we found a natural topological decomposition of the wreath product $\wr_{i=1}^{\infty} A_i$ into two isomorphic abelian free groups of rank ρ. For this, we introduced the notion of a homogeneous automorphism and the notion of a crack of an automorphism.

Definition 17 An automorphism $g \in Aut(X^*)$ is called homogeneous, if for every level of the tree X^* the vertex permutations $\sigma_{g,w}$ at the vertices w in this level coincide (but vertex permutations at the vertices in distinct levels may differ).

Definition 18 An automorphism $h \in Aut(X^*)$ is called a crack of an automorphism $g \in Aut(X^*)$ with a cracking path $u \in X^\omega$ if the portrait of h coincides with the portrait of g except the vertex permutations $\sigma_{h,w}$ at the vertices w which are prefixes of u, where these vertex permutations are all trivial.

A system consisting of a homogeneous automorphism and its crack can be described by a 2-state automaton $A = (\{a,b\}, X, \mathcal{R})$, in which the wreath recursions $\mathcal{R}_i$ $(i \geq 1)$ are of the form:

$$\mathcal{R}_i \colon \begin{cases} a_i = (a_{i+1}, a_{i+1}, \ldots, a_{i+1})\pi_i, \\ b_i = (b_{i+1}, a_{i+1}, \ldots, a_{i+1}), \end{cases}$$

where $\pi_i \in Sym(X_i)$ (above, we assume the orderings of the alphabets X_i for which the letters from the corresponding cracking path of the automorphism $\widetilde{b} \in Aut(X^*)$ are in the first positions).

Proposition 8 ([H3], Proposition 3.5) *Let $S \subseteq Aut(X^*)$ be a set of mutually coprime homogeneous automorphisms and for every $s \in S$ let $\check{s}$ denotes an arbitrary crack of s. If the vertex groups $V_{S,i} \subseteq Sym(X_i)$ $(i \geq 1)$ are all abelian and transitive, then the union $S \cup \{\check{s} \colon s \in S\}$ topologically generates the wreath product $\wr_{i=1}^{\infty} V_{S,i}$.*

To use the above claim for the wreath product $\wr_{i=1}^{\infty} A_i$, we constructed the set

$$S = \{a_1, \ldots, a_\rho\} \subseteq \wr_{i=1}^{\infty} A_i,$$

which satisfies the conditions in the following proposition:

Proposition 9 ([H3], Proposition 4.1) *There is a ρ-element set $S \subseteq \wr_{i=1}^{\infty} A_i$ of homogeneous automorphisms, for which:*

- *S consists of mutually coprime automorphisms,*
- *the vertex group $V_{S,i}$ coincides with A_i for every $i \geq 1$,*
- *the group generated by S is an abelian free group of rank ρ, that is $\langle S \rangle \simeq C_\infty^\rho$; in particular, the set S forms a basis for the group $\langle S \rangle$.*

Now, let us consider an arbitrary infinite word

$$u_0 := x_1 x_2 x_3 \ldots \in X^\omega$$

and for every $1 \leq k \leq \rho$ and $i \geq 1$ let $a_{k,i} \in Aut(X_{(i)}^*)$ be the section of the automorphism a_k at a word of length $i-1$ (it does not signify which word we choose as the automorphism a_k is homogeneous) and let $b_{k,i} \in Aut(X_{(i)}^*)$ be the crack of $a_{k,i}$ with the cracking path $x_i x_{i+1} x_{i+2} \ldots \in X_{(i)}^\omega$. In particular $a_{k,i}$ is a homogeneous mutually coprime automorphism of the tree $X_{(i)}^*$. Let us define the groups:

$$H_i := \langle S_{a,i} \rangle, \quad K_i := \langle S_{b,i} \rangle, \quad G_i := \langle S_{a,b,i} \rangle,$$

where

$$S_{a,i} := \{a_{1,i}, \ldots, a_{\rho,i}\}, \quad S_{b,i} := \{b_{1,i}, \ldots, b_{\rho,i}\}, \quad S_{a,b,i} := S_{a,i} \cup S_{b,i}.$$

By Proposition 8, we get that the set $S_{a,b,1}$ topologically generates the wreath product $\wr_{i=1}^{\infty} A_i$. The required decomposition into two abelian free subgroups is defined by the following partition: $S_{a,b,1} = S_{a,1} \cup S_{b,1}$. In section 5 in [H3], we have derived this property as a property of the group $G_1 = \langle S_{a,b,1} \rangle$ together with another algebraic and geometric properties (actually, we have derived them for all the groups $G_i = \langle S_{a,b,i} \rangle$, $i \geq 1$), obtaining the following theorem:

Theorem 18 ([H3], Theorem 5.1) *For every $i \geq 1$, we have:*

(i) G_i can be presented as a semidirect product $G_i = H_i^{G_i} \rtimes K_i = K_i^{G_i} \rtimes H_i$, where $H_i^{G_i}$ denotes the normal closure of the group H_i in G_i,

(ii) the semigroup generated by the set $S_{a,b,i}$ is a free product of the semigroups generated by the sets $S_{a,i}$ and $S_{b,i}$; in particular G_i is of exponential growth,

(iii) $G_i/G_i' \simeq C_\infty^{2\rho}$; in particular $d(G_i) = 2\rho$,

(iv) G_i is a torsion-free, weakly branch, not finitely presented and centerless group without non-abelian free subgroups.

For the proof, we consider the set $S_{a,b,1}$ as a set of states in the automaton $\mathcal{B} = (S_{a,b,1}, X, \mathcal{R})$ with the following system of wreath recursions $\mathcal{R} = (\mathcal{R}_i)_{i\geq 1}$:

$$\mathcal{R}_i\colon \begin{cases} a_{1,i} &= (a_{1,i+1}, a_{1,i+1}, \ldots, a_{1,i+1})\sigma_{1,i}, \\ a_{2,i} &= (a_{2,i+1}, a_{2,i+1}, \ldots, a_{2,i+1})\sigma_{1,i}, \\ \vdots & \vdots \\ a_{\rho,i} &= (a_{\rho,i+1}, a_{\rho,i+1}, \ldots, a_{\rho,i+1})\sigma_{\rho,i}, \\ b_{1,i} &= (b_{1,i+1}, a_{1,i+1}, \ldots, a_{1,i+1}), \\ b_{2,i} &= (b_{2,i+1}, a_{2,i+1}, \ldots, a_{2,i+1}), \\ \vdots & \vdots \\ b_{\rho,i} &= (b_{\rho,i+1}, a_{\rho,i+1}, \ldots, a_{\rho,i+1}), \end{cases} \tag{9}$$

where $\sigma_{k,i} := \sigma_{a_{k,i},\epsilon}$ is the vertex permutation of $a_{k,i}$ at the root of the corresponding tree. In particular for every $i \geq 1$, we have: $A_i = \langle \sigma_{k,i} \colon 1 \leq k \leq \rho \rangle$ and $G(\mathcal{B}_i) = G_i = \langle S_{a,b,i} \rangle$, where $\mathcal{B}_i$ is the i-th transition of the automaton $\mathcal{B}$. In order to derive the above properties, we use the combinatorial methods for the group-words $U \in F(S_{a,b,i})$ and their sections. To this aim, we introduce the notion of an (a,i)-syllable, as an arbitrary group-word $U \in F(S_{a,i})$, as well as the notion of a (b,i)-syllable, as a group-word $U \in F(S_{b,i})$. We call the syllable $U \in F(S_{a,i}) \cup F(S_{b,i})$ trivial, if it is an empty group-word ε, or it defines the neutral element in G_i. In particular, the group $H_i = \langle S_{a,i} \rangle$ consists of all elements defined by the (a,i)-syllables and the group $K_i = \langle S_{b,i} \rangle$ consists of all elements defined by the (b,i)-syllables.

Lemma 3 ([H3], Lemma 5.2) *For every $i \geq 1$, we have the isomorphisms $H_i \simeq K_i \simeq C_\infty^\rho$.*

Corollary 4 ([H3], Corollary 5.3) *The syllable $U \in F(S_{a,i})$ (resp. $U \in F(S_{b,i})$) is trivial if and only if for every $1 \leq k \leq \rho$ the sum of exponents on the letter $a_{k,i}$ (resp. on the letter $b_{k,i}$) in this syllable is equal to zero.*

For every group-word $U \in F(S_{a,b,i})$ there is $n \geq 1$ such that $U = W_1V_1 \ldots W_nV_n$, where each W_j is an (a,i)-syllable and each V_j is a (b,i)-syllable. If all the syllables W_j, V_j $(1 \leq j \leq n)$ are nontrivial (except, maybe, the case $W_1 = \varepsilon$ or $V_n = \varepsilon$), then U is called reduced. By using the wreath recursions (9), we define the section $U_{\{w\}} \in F(S_{a,b,i+|w|})$ at a word $w \in X_{(i)}^*$ in the same way as it was described in section 2.2, remembering additionally about deleting the trivial syllables at every time when the successive one-letter section appears. In particular $U_{\{w\}}$ is a reduced group-word. For a reduced group-word $U \in F(S_{a,b,i})$, we denote by $L(U)$ the number of all nontrivial (b,i)-syllables in this word. Directly by (9), we get: $L(U) \geq L(U_{\{w\}})$ for every

$w \in X^*_{(i)}$. Moreover, if $L(U) = L(U_{\{w\}})$, then $U_{\{w\}} = U_{(i+|w|)}$ (we use the notations from the section 2.2, that is the group-word $U_{(i+|w|)}$ is obtained from U by replacing each $a^{\eta}_{k,i}$ ($1 \leq k \leq \rho$, $\eta \in \{-1, 1\}$) with $a^{\eta}_{k,i+|w|}$ and each $b^{\eta}_{k,i}$ with $b^{\eta}_{k,i+|w|}$).

Lemma 4 ([H3], Lemma 5.5, Lemma 5.6) *If $U \in F(S_{a,b,i})$ is a reduced group-word and $L(U) > 1$, then there is $m \geq 1$ such that $L(U) > L(U_{\{w\}})$ for every $w \in X^m_{(i)}$; there is also $w \in X^*_{(i)}$ such that $L(U) = L(U_{\{w\}}) > L(U_{\{wx\}})$ for every $x \in X_{i+|w|}$. In particular, for every $U \in F(S_{a,b,i})$ there is $M \geq 1$ such that $L(U_{\{w\}}) \leq 1$ for all $w \in X^M_{(i)}$.*

By using induction on $L(U)$ together with Corollary 4 and Lemma 4, we derive the following proposition.

Proposition 10 ([H3], Proposition 5.7) *If $U \in F(S_{a,b,i})$ defines the neutral element in G_i, then for every letter from $S_{a,b,i}$ the sum of exponents on this letter in U is equal 0.*

By Proposition 10 and by Lemma 3, we obtain the isomorphism $G_i/G'_i \simeq C^{2\rho}_{\infty}$, which implies: $d(G_i) = 2\rho$. We also obtain the decomposition of G_i into the semidirect products ([H3], Theorem 5.1 (i)). For example, to show the equality $G_i = H_i^{G_i} \rtimes K_i$, we have $G_i = \langle H_i \cup K_i \rangle$, which implies that every element $g \in G_i$ is of the form $g'g''$ for some $g' \in H_i^{G_i}$ and $g'' \in K_i$. Thus, it is enough to show that the groups $H_i^{G_i}$ and K_i intersect trivially, but this follows from Proposition 10, as the group $H_i^{G_i}$ is generated by elements of the form $k^{-1}hk$ for $k \in K_i$ and $h \in H_i$.

To show that G_i is torsion-free ([H3], Theorem 5.1 (iv)), we consider an arbitrary group-word $U \in F(S_{a,b,i})$ which does not define the trivial automorphism. Let $w = vx_0$ ($v \in X^*_{(i)}$, $x_0 \in X_{i+|v|}$) be the shortest word for which $U(w) \neq w$. Denote $t := |v|$. For the section $U_{\{v\}} \in F(S_{a,b,i+t})$, we obviously have: $U_{\{v\}} = W_1V_1 \dots W_nV_n$, where each W_j is an $(a, i+t)$-syllable and each V_j is a $(b, i+t)$-syllable. Since $U(v) = v$ and $U(w) \neq w$, we have $U_{\{v\}}(x_0) \neq x_0$. By the recursions (9), we see that for all $1 \leq j \leq n$ and $x \in X_{i+t}$ the equality $V_j(x) = x$ holds. Thus, for the $(a, i+t)$-syllable $W := W_1 \dots W_n$, we have: $W(x_0) = U_{\{v\}}(x_0) \neq x_0$. Thus, by Corollary 4, there is a letter from $S_{a,i}$ such that the sum of exponents on this letter in the syllable W is nonzero. Consequently, the sum of exponents on this letter in the whole word $U_{\{v\}}$ is also nonzero. Hence, for any power $(U_{\{v\}})^m$ ($m \geq 1$) the sum of exponents on this letter in this power is nonzero. But, in view of the equality $U(v) = v$, we have: $(U^m)_{\{v\}} = \widehat{(U_{\{v\}})^m}$, that is the section $(U^m)_{\{v\}}$ is simply obtained from the power $(U_{\{v\}})^m$ by deleting the trivial syllables. Thus, there is a letter such that for every $m \geq 1$ the sum of exponents on this letter in the section $(U^m)_{\{v\}}$ is nonzero, which, by Proposition 10, implies that the section $(U^m)_{\{v\}}$, and consequently also the power U^m, does not define the trivial automorphism.

It follows by Proposition 10 that the empty word is the only reduced group-word U which defines the neutral element and satisfies the inequality $L(U) \leq 1$. Thus, by Lemma 4 and by the above observation that G_i is torsion-free, we obtain the following result.

Corollary 5 ([H3], Corollary 5.8) *If a group-word $U \in F(S_{a,b,i})$ defines the neutral element in G_i, then U is disappearing, that is there is $N \geq 0$ such that for every $w \in X^N_{(i)}$ the section $U_{\{w\}}$ is the empty group-word.*

Lemma 5 ([H3], Lemma 5.9) *There are group-words $U \in F(S_{a,b,i})$ defining the neutral element and having arbitrarily large depth.*

As a result, we obtain that the group G_i is not finitely presented ([H3], Theorem 5.1 (vi)). Next, from the following lemma, we obtain that G_i is centerless [H3], Theorem 5.1 (v)).

Lemma 6 ([H3], Lemma 5.10) *For arbitrary $U \in F(S_{a,b,i})$, $W \in F(S_{a,i})$ and $V \in F(S_{b,i})$, if the both group-words $UWU^{-1}W^{-1}$ and $UVU^{-1}V^{-1}$ define the neutral element in G_i, then at least one of the group-words U, W or V defines the neutral element, or is the empty group-word.*

To show that the group G_i does not contain non-abelian free subgroups ([H3], Theorem 5.1 (vii)), we can prove that the commutator subgroup G'_i is a nearly finitary group, and use Proposition 4 ([H6], Proposition 1). However, the original proof in [H3] uses the previously quoted Nekrashevych's alternative (Theorem 7) together with the amenability property concerning the components of the Schreier graphs arising from the action of groups generated by so-called almost finitary automorphisms ([61], Proposition 2.2).

Theorem 18 (iii) combined with the formula for the rank of the group $\wr_{i=1}^{\infty} A_i$ gives the natural construction of groups which solve the following problem (see also [H3], p. 1266):

Problem 1 How much the rank $d(G)$ of a group G generated by an automaton may differ from the rank $d(\overline{G})$ of its topological closure (in the automorphism group $Aut(X^*)$ of the corresponding tree X^*)?

Indeed, for the automaton $\mathcal{B}$ from our construction, we have $d(G(\mathcal{B})) = 2\rho$ and $\overline{G(\mathcal{B})} = \wr_{i=1}^{\infty} A_i$. If we now define the groups A_i as follows: $A_1 := C_{n_1}^n$, $A_i := C_{n_i}$ for $i \geq 2$, where $n > 1$ and $(n_i)_{i \geq 1}$ is an arbitrary sequence of pairwise coprime numbers $n_i > 1$, then we obtain the automaton from the following claim.

Corollary 6 *Let $n > 1$ be a natural number. Then there is a changing alphabet $X = (X_i)_{i \geq 1}$ and an automaton A over X such that the group $G := G(A) \leq Aut(X^*)$ generated by this automaton satisfies: $d(G) - d(\overline{G}) = n$.*

The above claim refers to the result due to G. A. Noskova ([64]), who investigated the differences $d(G) - d(\widehat{G})$ for abstract residually finite groups G.

Theorem 19 (Noskov) *For every $n \geq 1$ there is a finitely generated metabelian group G for which $d(G) - d(\widehat{G}) \geq n$.*

On the other hand, for polycyclic groups, we have the following result due to P. A. Linnel'a and D. Warhursta ([52]):

Theorem 20 (Linnel, Warhurst) $d(G) - d(\widehat{G}) \leq 1$ *for every polycyclic group G.*

The above two results suggest the further study of the group $G(\mathcal{B})$ in the direction of describing its finite index normal subgroups, which might allow to compare more efficiently the topological closure $\overline{G(\mathcal{B})}$ with the profinite completion $\widehat{G(\mathcal{B})}$. Obviously, we have $d(\overline{G(\mathcal{B})}) \leq d(\widehat{G(\mathcal{B})})$, as $\overline{G(\mathcal{B})}$ is a homomorphic image of $\widehat{G(\mathcal{B})}$. In particular, it would be interesting to know whether or not the group $G(\mathcal{B})$ has the congruence subgroup property. If so, then we would have the isomorphism $\overline{G(\mathcal{B})} \simeq \widehat{G(\mathcal{B})}$.

2.6 Amenability – paper [H8]

In 2016, K. Juschenko, V. Nekrashevych and M. de la Salle ([45]) formulated a new amenability condition for a wide class of groups acting by homeomorphisms on a topological space. As one of the applications, they obtained the following amenability condition for the class of groups generated by bounded automorphisms.

Theorem 21 (Juschenko, Nekrashevych, de la Salle, [45]) *Let $X = (X_i)_{i\geq 1}$ be a changing alphabet. If $G \leq Aut(X^*)$ is a group of bounded automorphisms for which the isotropy groups G_u ($u \in X^\omega$) are all amenable, then G is also amenable.*

The isotropy group G_u ($u \in X^\omega$) from the above theorem is defined as the quotient group $Stab_G(u)/N_{G,u}$, where $N_{G,u}$ consists of those automorphisms $g \in Stab_G(u)$ which act trivially on some neighborhood of u (i.e. on a set of the form $wX^*_{(|w|+1)}$ for some prefix $w \prec u$).

It was also shown in [45] that the above criterion implies the amenability for various known and previously studied groups generated by rooted and directed automorphisms, including the Brieussell's ([20]) constructions of dense subgroups in the wreath products of alternating groups, as well as the constructions of dense subgroups in the wreath products of abelian groups described in section 2.4 (including the special case studied by Fink). On the other hand, the Segal's construction of a dense subgroup in the wreath product $\wr_{i=1}^{\infty} PSL_2(p_i)$ gives te group which is not amenable.

The following natural problem arises: is it possible to use Theorem 21 to formulate an amenability criterion for groups generated by homogeneous automorphisms and their cracks? We have found such a criterion in the paper [H8]. To this aim, we introduced the following notion of singularity.

Definition 19 For a subgroup $G \leq Aut(X^*)$ and a word $u \in X^\omega$, we say that u is G-singular if for every $g \in G$ and every beginning $w \in X^*$ of u for which the section $g_{\{w\}}$ is non-trivial, the action of this section on the corresponding suffix of u is also non-trivial (i.e. if $u = wv$ and $g_{\{w\}} \neq id_{X^*_{(|w|+1)}}$, then $g_{\{w\}}(v) \neq v$).

The main result of the paper [H8] is the following theorem:

Theorem 22 ([H8], Theorem 2.6) *Let $\Gamma \leq Aut(X^*)$ be a group of homogeneous automorphisms and $u \in X^\omega$ be an arbitrary Γ-singular word. Let $\widetilde{\Gamma}$ be the group generated by the cracks of elements from Γ with the cracking path u. Then the group $G = \langle \Gamma \cup \widetilde{\Gamma} \rangle$ is amenable if and only if the group Γ is amenable.*

For the proof, we use the notion of a nearly finitary group, which we introduced in [H6]. Next, we derive the following result:

Proposition 11 ([H8], Proposition 3.4) *If $G \leq Aut(X^*)$ is a nearly finitary group, then all its isotropy groups G_u ($u \in X^\omega$) are trivial.*

Now, as a direct result of the above proposition and Theorem 21, we obtain:

Theorem 23 ([H8], Theorem 3.5) *If $G \leq Aut(X^*)$ is a nearly finitary group, then G is amenable.*

In one direction the criterion formulated in Theorem 22 directly follows from the known property that the class of amenable groups is closed under taking subgroups. For the converse, we constructed a nearly finitary normal subgroup $K \triangleleft G$ such that the quotient group G/K is a homomorphic image of Γ and use the property that the class of amenable groups is closed under taking extensions and homomorphic images. We also believe that Theorem 22 holds without the singularity assumption on the cracking path u; however, when dropping this assumption, we were not able to show that the constructed group K is a nearly finitary group.

We also used our criterion to the group

$$\mathcal{G} := G_1 = \langle S_{a,b,1} \rangle$$

constructed in the previous section, obtaining the following result (the same reasoning as in the proof of the below theorem gives the amenability of all groups $G_i = \langle S_{a,b,i} \rangle$, $i \geq 1$):

Theorem 24 ([H8], Theorem 2.7) *The group $\mathcal{G}$ is amenable.*

Theorem 23 can be used for the almost minimal automaton realization from [H6], obtaining the following result:

Corollary 7 *For an arbitrary wreath product $\wr_{i=1}^{\infty} H_i$ of non-abelian simple transitive permutation groups (H_i, X_i) on finite sets X_i there is an almost minimal automaton realization by a 3-state bounded automaton $\mathcal{A}$ over the alphabet $X = (X_i)_{i\geq 1}$. The group $G(\mathcal{A})$ generated by $\mathcal{A}$ is an amenable not finitely presented group of exponential growth, and its two generators corresponding to the nontrivial states of $\mathcal{A}$ generate a free semigroup; this is also an example of a self-replicating weakly branch and nearly finitary group.*

Remark 5 We suppose that the system (7) of wreath recursions for the automaton $\mathcal{A}$ from the above corollary defines for every $i \geq 1$ the isomorphism of the group $G(\mathcal{A}_i)$ with the wreath product $G(\mathcal{A}_{i+1}) \wr_{X_i} H_i$. In this case, we could use the recent results due to K. Juschenko ([43]) and obtain that $G(\mathcal{A})$ is an amenable group which is not elementary subexponentially amenable. Moreover, different sequences $(H_i, X_i)_{i\geq 1}$ probably give non-isomorphic automata groups. In consequence, we could obtain uncountable many such groups. Note that the first example of a group which is not elementary subexponentially amenable was the basilica group constructed in 2002 by Grigorchuk and Żuk ([38]). However, the proof that this is an amenable group was given in 2005 by L. Bartholdi and B. Virág – [9]. The basilica group is generated by a 3-state Mealy automaton $A = (\{a, b, id\}, \{0, 1\}, \mathcal{R})$ with the following system of wreath recursions:

$$\mathcal{R}\colon \begin{cases} a &= (id, b), \\ b &= (id, a)(0,1), \\ id &= (id, id). \end{cases}$$

2.7 The characterization of wreath products by automata – paper [H7]

Let $X = (X_i)_{i\geq 1}$ be an arbitrary changing alphabet. In 2010 Bondarenko ([15]) observed, that under some general conditions on a sequence $(G_i, X_i)_{i\geq 1}$ of transitive permutation groups, there is a finite set of rooted and directed automorphisms of the tree X^* which topologically generates the wreath product $W_\infty := \wr_{i=1}^{\infty} G_i$, obtaining in this way a criterion ([15], Theorem 1.1) when such a wreath product is topologically finitely generated. In spite of the fact that the full characterization of a finite topological generation for wreath products $\wr_{i=1}^{\infty} G_i$ was given by

E. Detomi and A. Lucchini in 2013 (Theorem 4), the detailed analysis of the Bondarenko's proof allows to get an insight into the construction of the corresponding topological generating set. This construction gave an idea, which finally allowed us to obtain the full characterization of the sequences $(G_i, X_i)_{i\geq 1}$ for which the wreath product $\wr_{i=1}^{\infty} G_i$ is generated by an automaton over the alphabet $X = (X_i)_{i\geq 1}$, as well as, in the case when X is fixed, by a Mealy automaton.

Theorem 25 ([H7], Theorem 1) *Let $(G_i, X_i)_{i\geq 1}$ be a sequence of transitive permutation groups.*

(i) Then the wreath product $W_\infty = \wr_{i=1}^{\infty} G_i$ is generated by an automaton over the alphabet $X = (X_i)_{i\geq 1}$ if and only if the following two conditions hold:

(a) the sequence $(d(G_i))_{i\geq 1}$ is bounded,

(b) $d(\prod_{i\geq 1} G_i/G_i') < \infty$.

(ii) If the alphabet $X = (X_i)_{i\geq 1}$ is fixed, then the wreath product W_∞ is generated by a Mealy automaton over X if and only if the following conditions hold:

(a') the sequence $(G_i)_{i\geq 1}$ is decreasing, i.e. $G_1 \geq G_2 \geq \dots$,

(b') the smallest group in this sequence is perfect.

If the alphabet $X = (X_i)_{i\geq 1}$ is bounded, then both the sequences $(d(G_i))_{i\geq 1}$ and $(d(G_i)/N_{i-1})_{i\geq 2}$ are bounded (we use notations as in Theorem 4). Consequently, we obtain:

Corollary 8 ([H7], Corollary 1) *Let $(G_i, X_i)_{i\geq 1}$ be a sequence of transitive permutation groups. If the alphabet $X = (X_i)_{i\geq 1}$ is bounded or the sequence $(d(G_i))_{i\geq 1}$ is bounded, then the following statements are equivalent*

(i) the wreath product $\wr_{i=1}^{\infty} G_i$ is generated by an automaton over X,

(ii) the wreath product $\wr_{i=1}^{\infty} G_i$ is topologically finitely generated.

If X is fixed, then the following statements are equivalent:

(iii) the wreath product $\wr_{i=1}^{\infty} G_i$ is generated by a Mealy automaton over X,

(iv) the wreath product $\wr_{i=1}^{\infty} G_i$ is topologically finitely generated and the sequence $(G_i)_{i\geq 1}$ is decreasing.

On the other hand, it is possible to choose an alphabet $X = (X_i)_{i\geq 1}$ and a sequence $(G_i, X_i)_{\geq 1}$ of transitive perfect groups such that the sequence $(d(G_i)/N_{i-1})_{i\geq 2}$ is bounded but the sequence $(d(G_i))_{i\geq 1}$ is unbounded ([15], Example 3.5). Consequently, we obtain:

Corollary 9 ([H7], Corollary 2) *There is a sequence $(G_i, X_i)_{i\geq 1}$ of transitive permutation groups on finite sets X_i such that the wreath product $\wr_{i=1}^{\infty} G_i$ is topologically finitely generated but there is no automaton which generates this wreath product.*

For the proof of the first part of Theorem 25, we introduced the notion of a generating basis of an arbitrary infinite sequence $(G_i)_{i\geq 1}$ of finite groups.

Definition 20 A generating basis (of degree m) of a sequence $(G_i)_{i\geq 1}$ of finite groups is a sequence $\Gamma = (\Gamma_i)_{i\geq 1}$ such that each Γ_i is an m-tuple $(g_{1i}, \dots, g_{mi})$ of elements of the group G_i and the following conditions hold:

- for each $i \geq 1$ the elements from Γ_i generate the group G_i, and
- there is $1 \leq k \leq m$ such that for each $i \geq 1$ the set $\{\overline{g_{1i}}, \ldots, \overline{g_{ki}}\}$ generates the abelianization G_i/G_i', and
- the elements $\overline{g_{ji}} \in G_i^{ab}$, $\overline{g_{ji'}} \in G_{i'}^{ab}$ are of coprime orders for all $1 \leq j \leq k$ and $i \neq i'$.

Proposition 12 ([H7], Proposition 3) *A sequence $(G_i)_{i\geq 1}$ of finite groups has a generating basis if and only if the conditions (a)–(b) from Theorem 25 are satisfied. Moreover, if these conditions are satisfied, then the sequence $(G_i)_{i\geq 1}$ has a generating basis of degree $m := d_1 + d_2$, where $d_1 := d(\prod_{i\geq 1} G_i/G_i')$, $d_2 := \max_{i\geq 1}(d(G_i))$.*

Proposition 13 ([H7], Proposition 4) *Let $(G_i, X_i)_{i\geq 1}$ be a sequence of transitive permutation groups satisfying the conditions (a)–(b) from Theorem 25. Let $((g_{1i}, \ldots, g_{mi}))_{i\geq 1}$ be an arbitrary generating basis of this sequence. Additionally, let us assume that for every $i \geq 1$ the action of the commutator subgroup G_i' on the set X_i satisfies the following condition: there are two letters $x_i, x_i' \in X_i$ in the same orbit but with the different stabilizers. For each $1 \leq j \leq m$ let us define two automorphisms $R_j, D_j \in \wr_{i=1}^{\infty} G_i$ by their vertex permutations $\sigma_{R_j,w}$, $\sigma_{D_j,w}$ ($w \in X^*$) as follows:*

$$\sigma_{R_j,w} := \begin{cases} g_{j1}, & w = \epsilon, \\ id_{X_{|w|+1}}, & w \neq \epsilon, \end{cases} \quad \sigma_{D_j,w} := \begin{cases} g_{j(i+1)}, & w = x_1 \ldots x_{i-1}x_i',\ i \geq 1, \\ id_{X_{|w|+1}}, & \text{otherwise.} \end{cases}$$

Then the set $\{R_1, \ldots, R_m, D_1, \ldots, D_m\}$ topologically generates the wreath product $\wr_{i=1}^{\infty} G_i$.

In particular, we see that each automorphism R_j from the above proposition is rooted, and each automorphism D_j is 1-directed with the direction $x_1x_2x_3\ldots \in X^\omega$. On basis of the above construction, we may define the *standard automaton* generating the wreath product $\wr_{i=1}^{\infty} G_i$. In such an automaton, the set of states consists of $3m$ states, such that m of these states define the automorphisms R_j, and the remaining $2m$ states define the automorphisms D_j. Unfortunately, such an automaton is not universal, as it can generate the wreath product $\wr_{i=1}^{\infty} G_i$ under the above described additional condition (beside the necessary conditions (a)–(b)) on the actions of the commutator subgroups G_i'. In order to obtain a universal construction, we observed that some modification of the standard automaton could be introduced. This led us to the automaton from the following proposition.

Proposition 14 ([H7], Proposition 10) *Let $(G_i, X_i)_{i\geq 1}$ be a sequence of transitive permutation groups satisfying the conditions (a)–(b) from Theorem 25. Let $((g_{1i}, \ldots, g_{mi}))_{i\geq 1}$ be an arbitrary generating basis of this sequence. For every $i \geq 1$ let us choose two letters $x_i, x_i' \in X_i$ and let us define an automaton $\mathcal{A} = (S, X, \varphi, \psi)$ in the following way:*

- $S = \{e\} \cup \bigcup_{j=1}^{m}\{r_{j0}, r_{j1}, r_{j2}, r_{j3}\} \cup \bigcup_{j=1}^{m}\{d_{j1}, d_{j2}, d_{j3}\}$,
- *the sequence $\varphi = (\varphi_i)_{i\geq 1}$ of transition functions $\varphi_i \colon S \times X_i \to S$ is defined as follows: $\varphi_i(e, x) = \varphi_i(r_{j0}, x) = e$ and*

$$\varphi_i(r_{j(s+1)}, x) = \begin{cases} r_{js}, & x = x_i', \\ e, & x \neq x_i', \end{cases}$$

$$\varphi_i(d_{j(s+1)}, x) = \begin{cases} d_{j(s+1)}, & x = x_i, \\ r_{j(s+1)}, & x = x_i',\ i \equiv -1 \pmod 3, \\ e, & \textit{otherwise}, \end{cases}$$

- *the sequence $\psi = (\psi_i)_{i\geq 1}$ of output functions $\psi_i\colon S \times X_i \to X_i$ is defined as follows:*

$$\psi_i(r_{j0}, x) = g_{ji}(x), \quad \psi_i(e, x) = \psi_i(d_{j(s+1)}, x) = \psi_i(r_{j(s+1)}, x) = x$$

for all $i \geq 1$, $x \in X_i$, $1 \leq j \leq m$, $s \in \{0, 1, 2\}$. Then the automaton $\mathcal{A}$ generates the wreath product $\wr_{i=1}^{\infty} G_i$.

Note at first that the group $G(\mathcal{A})$ generated by the above defined automaton $\mathcal{A}$ is a subgroup of the wreath product $\wr_{i=1}^{\infty} G_i$, as for every $i \geq 1$ the vertex group $V_{\mathcal{A},i} \leq Sym(X_i)$ (that is the group generated by the labels $\sigma_{s,i}\colon x \mapsto \psi_i(s, x)$ for $s \in S$) coincides with G_i. In order to obtain a clear description of the actions of the generators $\widetilde{s} \in G(\mathcal{A})$ ($s \in S$) on the tree X^*, and consequently to prove that $\mathcal{A}$ indeed generates the wreath product $\wr_{i=1}^{\infty} G_i$, we introduced the concept of ξ-partition A_ξ for an arbitrary automaton $A = (S, X, \varphi, \psi)$ and a strictly increasing sequence $\xi = (t_i)_{i\geq 0}$ of integers with $t_0 := 0$. Namely, we define the automaton A_ξ as an automaton

$$A_\xi := (S, X_\xi, \varphi_\xi, \psi_\xi)$$

in which the changing alphabet X_ξ is a so-called ξ-partition of the alphabet X, that is

$$X_\xi := (Y_i)_{i\geq 1}, \quad Y_i := \prod_{r=t_{i-1}+1}^{t_i} X_r, \quad i \geq 1,$$

and the transition functions $\varphi_{\xi,i}\colon S \times Y_i \to S$ and output functions $\psi_{\xi,i}\colon S \times Y_i \to Y_i$ are defined in such a way that in every moment $i \geq 1$ the automaton A_ξ, being in an arbitrary state $s \in S$ and reading from the input tape an arbitrary letter $y \in Y_i$ (which is a word over the alphabet $(X_j)_{j>t_{i-1}}$), imitates the behaviour of the automaton A, which being in a moment $t_{i-1} + 1$ in a state s, it reads from the input tape the word y. For the proof of Proposition 14, we now use the following lemma:

Lemma 7 ([H7], Lemma 1) *If for every $i \geq 1$ the equality holds $V_{A_\xi,i} = \wr_{r=t_{i-1}+1}^{t_i} V_{A,r}$ and additionally the automaton A_ξ generates the wreath product $\wr_{i=1}^{\infty} V_{A_\xi,i}$, then the automaton A generates the wreath product $\wr_{i=1}^{\infty} V_{A,i}$.*

Next, we consider the ξ-partition $\mathcal{A}_\xi = (S, Y, \varphi_\xi, \psi_\xi)$ of the automaton $\mathcal{A}$, where $\xi = (3i)_{i\geq 0}$, and we derive the description of transition and output functions in the automaton $\mathcal{A}_\xi$ ([H7], Proposition 9). On basis of this description, we show that the 3-iterated wreath product $H_i := \wr_{r=3i-2}^{3i} G_r$ ($i \geq 1$) is the vertex group $V_{\mathcal{A}_\xi,i}$ of the automaton $\mathcal{A}_\xi$, and, which is crucial, the condition from Proposition 13 for the actions of commutator subgroups H_i' on the sets $Y_i := X_{3i-2} \times X_{3i-1} \times X_{3i}$ is satisfied. We also construct the generating basis $(\hat{h}_{1,i}, \ldots, \hat{h}_{3m,i})_{i\geq 1}$ of degree $3m$ for the sequence $(H_i)_{i\geq 1}$. Next, we prove that the automorphisms defined by the states r_{js} and $d_{j(s+1)}$ ($1 \leq j \leq m$, $s = 0, 1, 2$) of the automaton $\mathcal{A}_\xi$ are, respectively, the rooted and 1-directed automorphisms of the tree X_ξ^*, and these automorphisms can be described by using the basis $(\hat{h}_{1,i}, \ldots, \hat{h}_{3m,i})_{i\geq 1}$ in the same way as the automorphisms R_j, D_j from Proposition 13 (for the corresponding basis). As a result, we obtain that the automaton $\mathcal{A}_\xi$ generates the wreath product $\wr_{i=1}^{\infty} V_{\mathcal{A}_\xi,i} = \wr_{i=1}^{\infty} H_i$ and hence, by Lemma 7, we get that the automaton $\mathcal{A}$ generates the wreath product $\wr_{i=1}^{\infty} G_i$.

We also see by above that only $6m$ states of the automaton $\mathcal{A}$ are used to generate $\wr_{i=1}^{\infty} G_i$. Consequently, we obtain the following estimation: $d(\wr_{i=1}^{\infty} G_i) \leq 6m$. Thus, the above construction provides the following upper bound for the rank of the wreath product $\wr_{i=1}^{\infty} G_i$, as well as for the number of states in an optimal automaton for this wreath product.

Corollary 10 ([H7], Corollary 3) *Let $(G_i, X_i)_{i\geq 1}$ be an arbitrary sequence of transitive permutation groups. Denote $m := d_1 + d_2$, where $d_1 := d(\prod_{i\geq 1} G_i/G_i')$ and $d_2 := \sup_i(d(G_i))$. Then $d(W_\infty) \leq 6m$, and the number of states in an optimal automaton for the group $\wr_{i=1}^{\infty} G_i$ is not greater than $7m + 1$.*

The converse part of Theorem 25 (i) follows from the observation that if the wreath product $\wr_{i=1}^{\infty} G_i$ is generated by an arbitrary automaton $A = (S, X, \varphi, \psi)$, then this wreath product, and consequently also the direct product $\prod_{i\geq 1} G_i/G_i'$ (as its homomorphic image), must be topologically finitely generated. Further, for every $i \geq 1$ the group G_i is generated by the vertex permutations $\sigma_{\widetilde{s},w} \in Sym(X_i)$ for $s \in S$ and $w \in X^{i-1}$. But the vertex permutation $\sigma_{\widetilde{s},w}$ ($s \in S$, $w \in X^{i-1}$) is the restriction of the section $\widetilde{s}_{\{w\}}$ to the set of one-letter words, and, in turn, this section is the transformation defined by a state of the automaton $A_i = (S, (X_j)_{j\geq i}, (\varphi_j)_{j\geq i}, (\psi_j)_{j\geq i})$. Thus, we have $d(G_i) \leq |S|$ for every $i \geq 1$.

To prove Theorem 25 (ii), we consider an arbitrary transitive perfect group (G, X) on a finite set X, and by using its arbitrary m-element generating set, we construct a $(7m+1)$-state Mealy automaton $\mathcal{B}$ over X. We next show that the automaton $\mathcal{B}$ generates the wreath power $P_\infty := \wr_{i=1}^{\infty} G^{(i)}$ of the group G ([H7], Proposition 12). To this aim, we study the automorphisms defined by the states of the automaton $\mathcal{B}_\xi$, where $\xi := (2i)_{i\geq 0}$ ([H7], Proposition 11). Finally, for an arbitrary sequence $(G_i, X)_{i\geq 1}$ of transitive groups satisfying the conditions (a')–(b') of Theorem 25, we use the construction of the automaton $\mathcal{B}$ together with the next lemma, to show that the wreath product $W_\infty = \wr_{i=1}^{\infty} G_i$ is generated by a Mealy automaton over X.

Lemma 8 ([H7], Lemma 2) *If $A = (S, X, \varphi, \psi)$ is a Mealy automaton over the alphabet X and (G, X) is a transitive permutation group such that the labels $\sigma_s \in Sym(X)$ ($s \in S$) of all the states belong to G, then the wreath product $G(A) \wr_X G$ is generated by a Mealy automaton over X. In particular, the group $\overline{G(A)} \wr_X G$ is generated by a Mealy automaton over X.*

For the converse, if the wreath product $W_\infty = \wr_{i=1}^{\infty} G_i$ is generated by a Mealy automaton $A = (S, X, \varphi, \psi)$, then the group G_i ($i \geq 1$) is generated by the set

$$\mathcal{S}_i := \{\sigma_{\widetilde{s},w} \colon s \in S, w \in X^{i-1}\}.$$

But, for all $i \geq 2$ and $w \in X^{i-1}$, we have: $w = xv$ for some $v \in X^{i-2}$ and $x \in X$, and hence, for every $s \in S$, we have: $\sigma_{\widetilde{s},w} = \sigma_{\widetilde{q},v}$, where $q := \varphi(s, x) \in S$, which implies $\mathcal{S}_i \subseteq \mathcal{S}_{i-1}$. Thus the sequence $(G_i)_{i\geq 1}$ is decreasing. The smallest group in this sequence (let us denote it by G_{i_0}) must be perfect, because otherwise the abelianization $K := G_{i_0}/G_{i_0}'$ would be a nontrivial abelian group and the infinite direct product $K^{\mathbb{N}}$ would be a homomorphic image of the wreath product W_∞ and hence, this wreath product would not be topologically finitely generated.

3 Discussion of other results

3.1 Some other achievements after PhD degree – papers [P1]–[P7]

The scientific achievements, obtained after the defence of my PhD Thesis in June 2005, include the following publications:

[P1] A. Woryna, *The generalized dihedral groups $Dih(C_\infty^n)$ as groups generated by time-varying automata*, ALGEBRA AND DISCRETE MATHEMATICS, 3 (2008), 98–111,

[P2] A. Woryna, *The group of balanced automorphisms of a spherically homogeneous rooted tree*, ANNALES MATHEMATICAE SILESIANAE, 23 (2009), 83–101,

[P3] A. Woryna, *The concept of duality for automata over a changing alphabet and generation of a free group by such automata*, THEORETICAL COMPUTER SCIENCE, 412 (45) (2011), 6420–6431; IF 0.665,

[P4] A. Woryna, *Automaton ranks of some self-similar groups*, LECTURE NOTES IN COMPUTER SCIENCE, 7183 (2012), 514–525,

[P5] A. Woryna, *The concept of self-similar automata over a changing alphabet and lamplighter groups generated by such automata*, THEORETICAL COMPUTER SCIENCE, 482 (2013), 96–110; IF 0.516,

[P6] A. Woryna, *The classification of abelian groups generated by time-varying automata and by Mealy automata over the binary alphabet*, INFORMATION AND COMPUTATION, 249 (2016), 18–27; IF 1.050,

[P7] A. Woryna, *On groups generated by bi-reversible automata: the two-state case over a changing alphabet*, JOURNAL OF COMPUTER AND SYSTEM SCIENCES, 86 (2017), 181–190; IF 1.678.

In the paper **[P2]**, we considered the group $Aut_c(X^*) \leq Aut(X^*)$ of cyclic automorphisms and its subgroup $J_c(X^*) \leq Aut_c(X^*)$ consisting of homogeneous automorphisms. The elements of the group $Aut_c(X^*)$ are automorphisms $g \in Aut(X^*)$ such that each vertex permutation $\sigma_{g,w}$ ($w \in X^*$) is a power of a fixed long cycle on the corresponding set of letters (the cycle is the same for all elements of the group $Aut_c(X^*)$). Next, for every homogeneous automorphism $g \in J_c(X^*)$, we replaced its vertex permutations at the vertices ending with an odd letter by their inverses (we have previously fixed the division of each set of letters into odd and even-indexed letters). The resulting cyclic automorphism (which obviously is no longer homogeneous), we called a balanced automorphism of the tree X^*. We showed that the subset $\mathcal{B} \subseteq Aut_c(X^*)$ of all balanced automorphisms forms a group if and only if for every $i \geq 1$ the implication holds: $2 \nmid n_i \Rightarrow n_{i+1} = 2$, where $n_i := |X_i|$. Depending on the sequence $(n_i)_{i\geq 1}$, we derived various algebraic properties of the group $\mathcal{B}$, obtaining a concrete realization of an uncountable family of uncountable metabelian groups satisfying the identity $x^2y^2 = y^2x^2$. For example, for the sequence $(2, n_2, 2, n_4, 2, n_5, \ldots)$ the group $\mathcal{B}$ is isomorphic to the infinite direct product $\prod_{i\geq 1} D_{2n_{2i}}$ of finite dihedral groups. Generally, the group $\mathcal{B}$ is residually nilpotent if and only if all the numbers n_i are powers of two. Moreover, the group $\mathcal{B}$ decomposes into the product $\mathcal{K}_0\mathcal{K}_1$ of its two abelian subgroups with the trivial intersection (however, in general, it is not a semi-direct product of abelian groups).

In the papers **[P1], [P3], [P5]**, we constructed and investigated the automaton realizations of some particular classes of groups, which are important and well-known in algebra. In **[P1]**, we studied the generalized dihedral groups $Dih(C_\infty^n)$ $(n \geq 1)$, which are semi-direct products $C_\infty^n \rtimes_\phi C_2$ with $\phi(0)$ the identity and $\phi(1)$ inversion. If we consider the power C_∞^n as a cubical lattice in the Euclidean space $\mathbb{R}^n$, then we may investigate the group $Dih(C_\infty^n)$ as a topologically discrete group of isometries of C_∞^n generated by translations and reflections in all points from C_∞^n. These groups constitute important examples of the so-called crystallographic groups. In the paper **[P1]**, we provided a new interpretation of the group $Dih(C_\infty^n)$, as a group $G(A)$ generated by an automaton A with a $(2n+2)$-element set of states $S = \{a_1, b_1, \ldots, a_{n+1}, b_{n+1}\}$ with the property that the automaton transformations $\widetilde{a_k}$ and $\widetilde{b_k}$ $(1 \leq k \leq n+1)$ are mutually inverse balanced automorphisms of order 2. In particular, we have $G(A) = \langle \widetilde{a_1}, \ldots, \widetilde{a_{n+1}} \rangle$. We provided the formula for the minimal length $||g||$ of any element $g \in G(A)$ (considered as a semi-group word on the letters $\widetilde{a_k}$, $1 \leq k \leq n+1$), obtaining the transparent algorithms solving both the word and conjugacy problem in $G(A)$. For the action of the group $G(A)$ on the tree X^*, we characterized the orbits of this action and the stabilizers. In particular, we obtained $Stab_{G(A)}(m) = Stab_{G(A)}(w) \simeq C_\infty^n$ and $Stab_{G(A)}(u) = \{id_{X^*}\}$ for all $m > 0$, $w \in X^m$, $u \in X^\omega$.

In **[P5]**, we constructed a universal automaton realization for the generalized dihedral group $KwrC_\infty := \bigoplus_{C_\infty} K \rtimes C_\infty$, where K is an arbitrary finitely generated abelian group (for the group $GwrC_\infty$ to be residually finite the group G must be abelian – [39]). For every finitely generated abelian group K, it is easy to construct an automaton A (over a changing alphabet) such that $G(A) \simeq K$. For this, we may use the construction of a diagonal automaton, that is an automaton $A = (S, X, \varphi, \psi)$ such that $\varphi_i(s, x) = s$ for all $i \geq 1$, $x \in X_i$ and $s \in S$. Along the way, we get the minimal automaton for K (i.e. an automaton in which the number of states equals $d(K)$). The main result of the paper **[P5]** is the observation that a simple manoeuvre on each transition function in a minimal diagonal automaton generating the direct product $K \times C_\infty$ leads to the automaton realization of the group $KwrC_\infty$. For this manoeuvre, in the i-th $(i \geq 1)$ transition function, we move from an arbitrary state to a fixed unique state (which is common for all $i \geq 1$) whenever the automaton reads from the input tape some fixed letter from the set X_i (the letter depending on a current state). This modification is universal, as it works for an arbitrary finitely generated (finite or infinite) abelian group K. Since $d(KwrC_\infty) = d(K \times C_\infty) = d(K) + 1$, we get a minimal automaton $A = (S, X, \varphi, \psi)$ for the group $KwrC_\infty$. We have also proved that this construction gives a self-similar automaton, that is for every $i \geq 1$ the map $\widetilde{s} \mapsto s_i$ for $s \in S$ induces an isomorphism $G(A) \simeq G(A_i)$, where $A_i = (S, (X_j)_{j \geq i}, (\varphi_j)_{j \geq i}, (\psi_j)_{j \geq i})$ is the i-th transition of the automaton A.

Presently, there are not known any realizations by a Mealy automaton of the group $KwrC_\infty$ with an infinite K. Beside of that, the only known minimal realization (by a Mealy automaton) concerns the simplest nontrivial case, that is $K = C_2$ ([37]). M. Kambites, P. Silva and B. Steinberg ([46, 72]) constructed also for an arbitrary finite abelian group K a single Mealy automaton A such that $G(A) \simeq KwrC_\infty$. This is a so-called reset automaton, in which both the set of states and the alphabet coincide with K. The previously minimal case $(K = C_2)$ deserves on a special attention, as the study of the group generated by the corresponding 2-state Mealy automaton allowed to find the counterexample to the strong Atiyach conjecture on the possible values of the so-called L^2-Betti numbers ([37]).

In the paper **[P3]**, we extended to arbitrary time-varying automata the notion of a dual automaton and its action on the free monoid S^* over the set of states. Previously, this notion was investigated only for Mealy-type automata. In **[P3]**, we used this extension to find an explicit and quite simple and accessible construction of a 2-state automaton A (so-called bireversible automaton) over an unbounded changing alphabet such that the group $G(A)$ is the non-abelian free group of rank 2. It is still an open problem if there exists a 2-state automaton over a bounded changing alphabet, which generates a non-abelian free group. In particular, we do not know, if there is a 2-state Mealy automaton generating a non-abelian free group. On the other hand, for every $n \geq 3$ there is a construction of an n-state Mealy automaton over the binary alphabet which generates F_n (non-abelian free group of rank n) ([73, 80]).

The problem of obtaining an explicit automaton realization of a non-abelian free group by Mealy automata (i.e. not necessarily as a group $G(A)$ generated by a single automaton, but as a subgroup $G \leq \mathcal{MA}(X^*)$ of all transformations defined by Mealy automata over X) turned out to be far from trivial. In 1983, Aleshin ([2]) constructed two transformations over the binary alphabet: one transformation defined by a 3-state Mealy automaton, and the second by a 5-state Mealy automaton, claiming that they generate F_2. But some mistakes have been found in his proof. The first automaton realization of F_2, presented in 1998 by A. M. Brunner and S. Sidki ([21]), was based on an embedding of the full linear group $GL_2(C_\infty)$ into the group $\mathcal{MA}(X^*)$ of the automaton transformations over the alphabet $X = \{1, 2, 3, 4\}$. In 2000, A. Oliinyk and V. Sushchansky ([65]) investigated the group of infinite unitriangular matrices over an arbitrary finite field $\mathbb{F}$ as a subgroup of the group $\mathcal{MA}(\mathbb{F}^*)$, which allowed them to construct an automaton realization of F_2 by Mealy automata over the binary alphabet (see also [41]). The problem of obtaining a concrete realization of a non-abelian free group by a single Mealy automaton turned out to be even more difficult. Sidki ([71]) conjectured that the 3-state Mealy automaton from the Aleshin's construction generates F_3. Grigorchuk and Żuk tried to confirm this, but the correct proof was presented by the siblings M. Vorobets and Ya. Vorobets ([79]) in 2007. It is worth to note that the above difficulties contrast with the Bhattacharjee's ([13]) result from 1995, according to which for every alphabet $X = (X_i)_{i \geq 1}$ and every $n \geq 1$ a random choice of an n-element sequence of automorphisms of the tree X^* almost surely gives a basis for F_n, that is the set of those n-tuples which do not satisfy this condition has measure zero (in reference to the Haar measure on the group $Aut(X^*)$).

In the paper **[P7]**, we introduced the notion of reversibility and bireversibility for automata over a changing alphabet. For Mealy-type automata, this notion was introduced by O. Macedońska, V. Sushchanky and V. Nekrashevych ([56]) in 2000, where it was observed that the automata group $Bir(X^*)$ consisting of the transformations defined by bireversible Mealy automata over the alphabet X is a dense subgroup in the group $\mathcal{MA}(X^*)$ of all transformations defined by Mealy automata over X, as well as that the group $Bir(X^*)$ is contained in the group $Comm(F_n)$ of virtual automorphisms of the free group F_n of rank $n := |X|$ (the Commensurateur of F_n) as a subgroup of the so-called vp-automorphisms. In 2005 Y. Glasner and S. Mozes ([27]) associate with any bireversible Mealy automaton a square complex together with its universal covering, which allowed to construct the first examples of automata A for which the group $G(A)$ is a non-abelian free group. Up until now, this is the only known construction of Mealy automaton which generates a non-abelian free group. Recently, I. Bondarenko, D. D'Angeli and E. Rodaro ([16]) constructed the first example of a bireversible Mealy automaton which generates not finitely presented group (isomorphic to $C_3 wr C_\infty$). Besides, I. Klimann ([50]) showed that the semigroup generated by an arbitrary 2-state reversible Mealy automaton is either finite or free. Whereas T. Godina and I. Klimann ([29]) proved that the connected reversible Mealy automata with a p-element set of states for a prime $p \geq 3$ can not generate an infinite torsion group.

Recall that a Mealy automaton $A = (S, X, \varphi, \psi)$ is called reversible if for every letter $x \in X$ the map $S \ni s \mapsto \varphi(s, x) \in S$ is a permutation of the set of states. If both the automaton

A and its inverse (denoted by A^{-1}) are reversible, then A is called bireversible. In **[P7]**, we extended these notions in a natural way to automata over a changing alphabet. Namely, we call an automaton $A = (S, X, \varphi, \psi)$ reversible if for all $i \geq 1$ and $x \in X_i$ the transformation $S \ni s \mapsto \varphi_i(s, x) \in S$ is a permutation of the set of states. If additionally, the inverse automaton A^{-1} is also reversible, then A is called bireversible.

As an example of a bireversible automaton, we may take an arbitrary diagonal automaton. Obviously, every diagonal Mealy automaton generates a finite group (which is a subgroup in the symmetric group on the corresponding alphabet). Also, every group generated by a diagonal automaton over a bounded changing alphabet is finite. The situation changes in the case of an unbounded alphabet. In **[P5]**, we have shown in this case that diagonal self-similar automata provide a universal method for defining arbitrary finitely generated residually finite groups. Namely, for every such a group G and an unbounded alphabet $X = (X_i)_{i\geq 1}$, there is a self-similar diagonal automaton $A = (S, X, \varphi, \psi)$ such that $G(A) \simeq G$ and $|S| = d(G)$. However, our proof was highly un-constructive. Namely, in spite of the fact that we had no idea for a given abstract group G how to explicitly construct the sequence $\psi = (\psi)_{i\geq 1}$ of output functions in the corresponding automaton, we were able to show directly from residual finiteness of G that such a construction was feasible.

In **[P3]**, we investigated bireversible automata of another type. These were 2-state automata $A = (\{a, b\}, X, \mathcal{R})$ with the following wreath recursions in the i-th transition ($i \geq 1$): $a_i = (b_{i+1}, a_{i+1}, \ldots, a_{i+1})\pi_i$, $b_i = (a_{i+1}, b_{i+1}, \ldots, b_{i+1})\tau_i$, where the permutations $\pi_i, \tau_i \in Sym(X_i)$ form a standard generating set of $Sym(X_i)$, that is τ_i is an arbitrary transposition and π_i is an arbitrary long cycle such that the letters $x_{1,i}$ and $x_{2,i}$ from the first two positions of the alphabet X_i satisfy: $\tau_i(x_{1,i}) = \pi_i(x_{1,i}) = x_{2,i}$. In the paper **[P3]**, we proved that if the alphabet $X = (X_i)_{i\geq 1}$ is unbounded and the sequence $(|X_i|)_{i\geq 1}$ is non-decreasing, then $G(A) \simeq F_2$, and in **[P7]**, we showed that if $|X_i| \geq 3$ for every $i \geq 1$ and $GCD(|X_i| - 1, |X_{i'}| - 1) = 1$ for $i \neq i'$, then the action of $G(A)$ on the tree X^* is spherically transitive.

A slight modification of the above wreath recursions gives the following recursions $a_i = (a_{i+1}, \ldots, a_{i+1})\pi_i$, $b_i = (b_{i+1}, \ldots, b_{i+1})\tau_i$ for $i \geq 1$. The group G generated by the obtained 2-state diagonal automaton is isomorphic to the subgroup $\langle \pi, \tau \rangle \leq \prod_{i\geq 1} Sym(X_i)$, where $\pi := (\pi_i)_{i\geq 1}$, $\tau := (\tau_i)_{i\geq 1}$. The case $X_i := \{1, \ldots, i+1\}$, $\pi_i := (1, 2, \ldots, i+1)$, $\tau_i := (1, 2)$ was investigated in [53, 54] and in [D3] (the description of the paper [D3] is on p. 47). In [D3], we have shown that the commutator subgroup G' is locally finite, the semigroup generated by π and τ is free, but G does not contain non-abelian free subgroups. It even turns out that G is amenable ([53], Example 4.1). Moreover, if we denote $G_I := \langle \pi_I, \tau_I \rangle$ for every $I \subseteq \mathbb{N}$, where π_I is obtained from π by replacing π_i for $i \in I$ with the trivial permutations, then for all $I, I' \subseteq \mathbb{N} \setminus \{1, 2, 3, 4\}$, we obtain that $I \neq I'$ implies that the groups G_I and $G_{I'}$ are not isomorphic ([54], Proposition 4.1). Consequently, we obtain uncountably many pairwise non-isomorphic groups $G(A)$ generated by a 2-state diagonal automaton A over the alphabet $X = (X_i)_{i\geq 1}$.

One of the main results in **[P7]** was to show that for an arbitrary changing alphabet $X = (X_i)_{i\geq 1}$ the following two statements are equivalent: (i) there is a 2-state bireversible automaton A over X for which $G(A) \simeq F_2$, (ii) the alphabet X is unbounded. In the proof of the above equivalency, we have shown that if the alphabet X is bounded, then for every 2-state bireversible automaton A over X the group $G(A)$ cannot be torsion-free groups. In particular, there is no 2-state biereversible Mealy automaton generating F_2. Further, on basis of the construction of an automaton from **[P3]**, we also obtain an explicit and clear construction of a 2-state automaton over an arbitrary unbounded changing alphabet which generates F_2.

In **[P7]**, we also classify all groups $G(A)$ generated by a 2-state bireversible automaton A over the binary alphabet (obviously, there are uncountable many of such automata). It turns out that all these groups are abelian and finite; and there are five such groups: the trivial group, C_2, $C_2 \times C_2$, C_4 and $C_2 \times C_4$. Only the first three of them are generated by the corresponding Mealy automaton. We also investigated the class $IR_{2,2}$ of groups $G(A)$ generated by a 2-state reversible automaton A over the binary alphabet, as well as the class $BIR_{2,3}$ of groups $G(A)$ generated by a 2-state bireversible automaton over the ternary alphabet $X = \{1, 2, 3\}$. We have shown that each of these classes contains infinitely many pairwise non-isomorphic finite groups.

In the paper **[P4]**, we introduced for any $m \geq 2$ and any abstract group G the notion of an automaton rank $ar(G, m)$ as a minimal number of states in a Mealy automaton A over an m-litter alphabet for which the isomorphism $G(A) \simeq G$ holds. Obviously, we have: $ar(G, m) \geq d(G)$ (if there is no such automaton, then we assume $ar(G, m) := \infty$). For example, the presently known automaton realizations of non-abelian free groups F_n ($n \geq 3$) give the equality $ar(F_n, 2) = d(F_n) = n$ for every $n \geq 3$. On the other hand, the exact value of $ar(F_2, m)$ ($m \geq 2$) is not known, but the following estimation $4 \leq ar(F_2, 2) \leq 6$ is true. By using the automaton realizations of the Baumslag-Solitar groups $BS(1, k) := \langle a, t\colon tat^{-1} = a^k \rangle$, derived by L. Bartholdi and Z. Šunića ([8]), we easily concluded in **[P4]** that for any $m, n \geq 2$ there is a group G with $d(G) = 2$ and $n \leq ar(G, m) < \infty$.

In **[P4]**, we constructed for all $n \geq 1$ and $m \geq 2$ an optimal Mealy automaton over an m-letter alphabet, which generates the abelian free group C_∞^n, obtaining in this way the automaton ranks of abelian free groups. It turned out that the optimal construction not always gives the minimal automaton (that is an automaton with an n-element set of states) – in the cases $m = 2$ or $n = 1$ the optimal automaton has $n + 1$ states. To this aim, we used the result due to Nekrashevych and Sidki ([62]), according to which every self-similar abelian free group of automorphisms of the tree $\{0, 1\}^*$ must be contracting.

We also suggest a wider approach by introducing for every abstract group G the automaton spectrum $sa(G)$ as the set of all pairs $(n, m) \in \mathbb{N} \times \mathbb{N}$ such that the isomorphism $G(A) \simeq G$ holds for some n-state Mealy automaton A over an m-letter alphabet. If $(n, m) \in sa(G)$, then obviously $(n', m') \in sa(G)$ for any integers $n' \geq n$, $m' \geq m$. Thus, if $[n] := \{n, n+1, \ldots\}$, then we obtain: $sa(G) = \emptyset$ or there is a unique number $k \in \mathbb{N}$ (I called this number the automaton width of G) such that $sa(G) = ([n_1] \times [m_1]) \cup \ldots \cup ([n_k] \times [m_k])$ for some uniquely defined sequences $(n_i)_{1 \leq i \leq k}$, $(m_i)_{1 \leq i \leq k}$ of natural numbers such that the first sequence is strictly increasing and the second – strictly decreasing. In **[P4]**, we have shown the equalities $sa(C_\infty) = [2] \times [2]$ and $sa(C_\infty^n) = ([n] \times [3]) \cup ([n+1] \times [2])$ for $n > 1$. We are able to determine the automaton spectra for the remaining homocyclic groups, as well as for some other finite abelian groups (paper in preparation). For example, for all $n \geq 1$ and $m \geq 2$, we have $sa(C_m^n) = [n] \times [\gamma(m)]$, where the number $\gamma(m)$ is defined as follows: if $m = p_1^{\mu_1} p_2^{\mu_2} \ldots p_t^{\mu_t}$ is the canonical decomposition of m, then $\gamma(m) = p_1^{\mu_1} + p_2^{\mu_2} + \ldots + p_t^{\mu_t}$ (in particular, assuming $\gamma(1) := 0$, we get the additive map $\gamma\colon \mathbb{N} \to \mathbb{N}$). Further, if G is a finite abelian group with the Shmitt canonical decomposition $G \simeq C_{m_1}^{n_1} \times C_{m_1 m_2}^{n_2} \ldots \times C_{m_1 \ldots m_k}^{n_k}$ ($n_i \geq 1$, $m_i \geq 2$), then by denoting $\gamma(G) := \gamma(m_1) + \ldots + \gamma(m_k)$, we obtain $(d(G)+1, \gamma(G)) \in sa(G)$, and if $n_1 > 1$, then $(d(G), \gamma(G)) \in sa(G)$; finally if $GCD(m_i, m_{i'}) = 1$ for $i \neq i'$, then $sa(G) = [d(G)] \times [\gamma(G)]$. In particular, we would like to know if there is a group G with the automaton width greater than 2. We suppose to find such groups among finite abelian groups for which the equality $m_1 = m_2 = \ldots = m_k$ holds in the above decomposition.

In the paper **[P6]**, for every natural number $n \geq 1$ and every abelian group G, we have shown that the isomorphism $G \simeq G(A)$ holds for some n-state time-varying automaton A over the binary alphabet $X = \{0, 1\}$ if and only if $d(G) \leq n$ and the torsion part of G is a 2-group. Moreover, for every such a group G, we provided an explicit construction of the corresponding automaton. As a byproduct, we obtained that there are infinitely many pairwise non-isomorphic groups of the form $G(A)$, where A is a 2-state automaton over the binary alphabet (we remember that the 2-state Mealy automata over the binary alphabet generate only six pairwise non-isomorphic groups). In **[P6]**, we separately investigated the case of Mealy automata, and we obtain an analogous characterization of abelian groups generated by an n-state Mealy automaton over the binary alphabet. In particular, there are exactly $2n$ groups of this form (up to isomorphism), and each of them is an elementary 2-group or an abelian free group. Note that the number of all n-state Mealy automata (up to isomorphism of automata) over the binary alphabet is finite and equal to $2^n \cdot n^{2n}$.

Obviously, we may investigate for arbitrary integers $m, n \geq 1$ the class $\mathcal{GTV}(n, m)$ of groups (up to isomorphism) of the form $G(A)$, where A is an n-state automaton over an m-letter alphabet. For every $n \geq 1$ the class $\mathcal{GTV}(n, 1)$ is trivial, and for $m \geq 1$ the clas $\mathcal{GTV}(1, m)$ is finite and consists of some finite cyclic groups. Thus, the smallest interesting case is the class $\mathcal{GTV}(2, 2)$, which, by above, is infinite. But, on the other hand, there are uncountably many (up to isomorphism of automata) 2-state automata over the binary alphabet. Hence, it would be interesting to know if the class $\mathcal{GTV}(2, 2)$ is uncountable. What other groups (besides abelian groups) belong to this class? In particular, does a non-abelian free group belong to this class? And maybe every 2-generated residually 2-group belongs to this class? Does this class contain an example of a group with an unsolvable word problem? As for the last question, it is even not known if there is an automaton A over a bounded alphabet such that the group $G(A)$ has an unsolvable word problem (remember that this problem is solvable in the class of groups generated by a Mealy automaton). On the other hand, there is a constructive example of a 2-state diagonal automaton A over an unbounded changing alphabet such that the group $G(A)$ has an unsolvable word problem. This example is based on the construction of a 2-generated residually finite group obtained in 2009 by G. Baumslaga and Ch. F. Millera III ([11]) on basis of the B. H. Neumann's groups from 1937.

3.2 The scientific achievements in the PhD Thesis – papers [D1]–[D5]

I defended my PhD Thesis, entitled "*Time-varying Mealy automata and groups generated by these automata*", in June 2005 at the Institute of Mathematics of the University of Silesia. My PhD supervisor was Professor Vitaliy Ivanovich Sushchansky. The scientific achievements obtained in this PhD project consist of the following publications:

[D1] A. Woryna, *Odwzorowania określone za pomocą automatów Mealy'ego zmiennych w czasie*, Zeszyty Naukowe Politechniki Śląskiej, Seria: Automatyka, z. 138 (2003), 201–215,

[D2] A. Woryna, *On representation of a semi-direct product of cyclic groups by a 2-state time-varying Mealy automaton*, Zeszyty Naukowe Politechniki Śląskiej, Seria: Matematyka-Fizyka, z. 91 (2004), 343–354,

[D3] A. Woryna, *On permutation groups generated by time-varying Mealy automata*, Publicationes Mathematicae, Debrecen, 67 (1-2) (2005), 115–130; IF 0.238,

[D4] A. Woryna, *Representations of a free group of rank two by time-varying Mealy-automaton*, Discussiones Mathematicae General Algebra and Applications, 25 (2005), 119–134,

[D5] A. Woryna, *On generation of wreath products of cyclic groups by two state time varying mealy automata*, International Journal of Algebra and Computation, 16 (2) (2006), 398–415; IF 0.357.

In the papers **[D1]–[D5]**, we considered an even wider class of automata over a changing alphabet, by allowing to change the sets of states in the consecutive moments of their actions. We called them time-varying Mealy automata. We formally defined such an automaton as a quintuple $A = (S, X, Y, \varphi, \psi)$, where $(S_i)_{i\geq 1}$ is a sequence of sets of states, $X = (X_i)_{i\geq 1}$ and $Y = (Y_i)_{i\geq 1}$ are changing alphabets, input and output, respectively, $\varphi = (\varphi_i)_{i\geq 1}$ is a sequence of transition functions of the form $\varphi_i\colon S_i \times X_i \to S_{i+1}$ and $\psi = (\psi_i)_{i\geq 1}$ is a sequence of output functions $\psi_i\colon S_i \times X_i \to Y_i$. For every state $s \in S_1$, we defined, analogically as for the automaton from Definition 2, the automaton function $\widetilde{s}\colon X^* \to Y^*$.

In **[D1]**, we only investigated the structural properties of these automata; we identified some natural subclasses, such as the class of automata with a fixed set of states, the class of periodic automata or the class of permutational automata. We introduced the notion of equivalency for such automata, and used this to compare various models of automata, identifying automata with a simpler structure. We characterized the automaton functions $f\colon X^* \to Y^*$, as well as we characterized the transformations $f\colon X^* \to X^*$ defined by permutational automata as those permutations of the tree X^* which preserve the beginnings and the lengths of words. We also used the notion of a section of an automaton function to characterize the functions defined by periodic automata.

In **[D3]**, we introduced the group $\mathcal{GA}(X^*)$ of all transformations defined by permutational time-varying Mealy automata over an input-output alphabet $X = (X_i)_{i\geq 1}$. We described a natural isomorphism of this group with the wreath product $\wr_{i=1}^{\infty} Sym(X_i)$. We introduced the notion of a group $G(A) = \langle \widetilde{s}\colon s \in S_1\rangle$ generated by a single permutational automaton $A = (S, X, X, \varphi, \psi)$. We showed that every n-generated ($n \geq 1$) residually finite group G is isomorphic to the group $G(A)$ generated by an n-state automaton A. For every $n \geq 1$, we constructed and investigated the 2-state automaton realization for the group which is dual to the lamplighter group $C_n wr C_\infty$, that is for the regular wreath product $C_\infty wr C_n$. We also considered the 2-state diagonal automaton A over the alphabet $X = (X_i)_{i\geq 1}$ in which $X_i := \{1, \ldots, i+1\}$, and the labels of the states in their i-th transitions ($i \geq 1$) form the standard generating set of the symmetric group $Sym(X_i)$, that is the transposition $\alpha_i := (1, 2)$ and the cycle $\beta_i := (1, 2, \ldots, i + 1)$. For the group $G := G(A)$ generated by this automaton, we showed that its action on the tree X^* is spherically transitive, as well as that this group contains an isomorphic copy of every finite group, it does not contain non-abelian free subgroups, but the semigroup generated by the two automaton generators is free. It was also shown that the commutator group G' is locally finite and the abelianization G/G' is isomorphic to the direct product $C_2 \times C_\infty$.

In **[D2]**, we considered the so-called Q-adic adding machine ([10]), where $Q = (n_i)_{i\geq 1}$ is an arbitrary sequence of natural numbers greater than one. We can identify such a machine with the cyclic automorphism $a \in Aut(X^*)$ defined by the following sequence of wreath recursions: $a_i = (id_{i+1}, \ldots, id_{i+1}, a_{i+1})\sigma_i$ for $i \geq 1$, where $id_i := id_{X^*_{(i)}}$ and $\sigma_i \in Sym(X_i)$ is a long cycle. The automorphism a is often useful (especially in the regular case) for various automaton realizations, and its properties are still investigated. For example, it can be shown that an arbitrary automorphism $g \in Aut(X^*)$ is a conjugate of a if and only if the cyclic group $\langle g \rangle$ acts spherically transitively on the tree X^* ([10]). We see that a is defined by an automaton with two states, and this is the minimal number of states to define this automorphism. In **[D2]**, we wanted to know what happens if we join the adding machine with some another nontrivial cyclic automorphism $b \in Aut(X^*)$ which interferes as little as possible in the automaton structure of a, that is it does not add new states within the automaton structure of the set $\{a, b\}$ and does not change the transition functions associated with a. We were interested in the group generated by the resulting 2-element set of cyclic automorphisms. One of such simplest "minimal interfering" automorphisms is the automorphism b with the following wreath recursion: $b = (a_2, id_2, \ldots, id_2)\sigma_1$. In **[D2]**, we showed that the group $G(A) := \langle a, b \rangle$ generated by the resulting 2-state automaton A is isomorphic to the subgroup $K := W' \cdot C_\infty$ of the semi-direct product $W := C_\infty^{n_1} \rtimes C_\infty$ with the action of C_∞ on $C_\infty^{n_1}$ by the cyclic left shift. The group K is an example of a torsion-free metabelian group, which is not nilpotent, its center is isomorphic to C_∞, the commutator subgroup is isomorphic to $C_\infty^{n_1 - 1}$, and the abelianization is isomorphic to $C_\infty \times C_{n_1}$. In particular, this construction describes a spherically transitive action of the group K on the tree X^*. In **[D2]**, we characterized the stabilizers $Stab_{G(A)}(w)$, $Stab_{G(A)}(m)$, $Stab_{G(A)}(u)$ ($w \in X^*$, $u \in X^\omega$, $m > 0$) and the orbits of the action on the set X^ω of infinite words. We also derived the presentation of K with reference to the generating set $\{a, b\}$.

In **[D4]**, we derived two distinct automaton realizations for the non-abelian free group F_2. In the first one, we used a 2-state diagonal automaton $A = (\{s_1, s_2\}, X, X, \varphi, \psi)$ over the alphabet $X = (X_i)_{i \geq 1}$ with $X_i = \{1, \ldots, i\}$ for $i \geq 1$. We discovered that the output functions in such an automaton can be defined on basis of the lexicographical ordering $\preceq$ of the set of all reduced group-words on two letters a and b, starting from $\epsilon \preceq a \preceq a^{-1} \preceq b \preceq b^{-1}$. To this aim, we constructed two permutations $a, b \in Sym(\mathbb{N})$ with the property that if $W = W(a, b)$ is the group-word from the n-th position ($n \geq 1$), then for the composition $\mathcal{W} \in Sym(\mathbb{N})$ corresponding to W, we have $\mathcal{W}(1) = n$. It was proved that the required output functions $\psi_i\colon \{s_1, s_2\} \times X_i \to X_i$ can be defined as follows: $\psi_i(s_1, x) = a(x)$ for $x \in a^{-1}(X_i)$, $\psi_i(s_1, x) = a_i(x)$ dla $x \notin a^{-1}(X_i)$, $\psi_i(s_2, x) = b(x)$ for $x \in b^{-1}(X_i)$, $\psi_i(s_2, x) = b_i(x)$ for $x \notin b^{-1}(X_i)$, where $a_i\colon X_i \setminus a^{-1}(X_i) \to X_i \setminus a(X_i)$ and $b_i(x)\colon X_i \setminus b^{-1}(X_i) \to X_i \setminus b(X_i)$ are arbitrary bijections. We also provided an explicit analytic description of the permutations a and b. However, they turned out to be quite complicated, which implied that this realization, although explicit and diagonal, did not provide a visible opportunity for the further study of the group $G(A)$ and its action on the tree X^*.

The second automaton realization of F_2 described in **[D4]** was based on the automaton over the alphabet $X = (X_i)_{i \geq 1}$ with $X_i = \{1, 2, \ldots i+2\}$ for $i \geq 1$. By that construction, we realized that the output functions in an automaton realization of F_2 can be defined in a quite simple way. Namely, the labels of the states in their i-th transition can be the successive powers of the cycle $(2, 3, \ldots, i+2)$. However, when defining the transition functions, it was necessary to add new states in the consecutive transitions of the automaton, such that in the i-th transition we needed $i+2$ states. In particular, that construction did not provide an automaton with a fixed set of states.

Te previously mentioned paper **[D5]** was the last, but also the most interesting result within my PhD Thesis. In that work, we constructed a 2-state permutational automaton $A = (\{a, b\}, X, X, \mathcal{R})$ with the sequence $\mathcal{R} = (\mathcal{R})_{i \geq 1}$ of wreath recursions, in which the system $\mathcal{R}_i$ consists of the following two wreath recursions: $a_i = (b_{i+1}, a_{i+1}, \ldots, a_{i+1})$, $b_i = (a_{i+1}, \ldots, a_{i+1})\sigma_i$, where the permutation $\sigma_i \in Sym(X_i)$ is an arbitrary long cycle. The main result was the observation that if $GCD(n_i, n_{i'}) = 1$ for all $i \neq i'$, then the automaton A generates the wreath product $\wr_{i=1}^{\infty} C_{n_i}$. For the group $G := G(A)$, we also derived the following properties (in the publication **[D5]**, we have additionally observed that G is weakly branch): G is a torsion-free not finitely presented centerless group, it acts spherically transitively on the tree X^*, the semigroup generated by the automaton generators is free, and the abelianization G/G' is isomorphic to $C_\infty \times C_\infty$.

3.3 Some results from outside group theory – papers [S1]–[S5]

In the following works, we studied some problems from outside group theory.

[S1] A. Woryna, *Liczby Stirlinga, a skoki narciarskie*, DELTA, 11 (2001), 6–7,

[S2] A. Woryna, *The solution of a generalized secretary problem via analytic expressions*, JOURNAL OF COMBINATORIAL OPTIMIZATION, 33 (4) (2017), 1469–1491; IF 1.235,

[S3] A. Woryna, *On the set of uniquely decodable codes with a given sequence of code word lengths*, DISCRETE MATHEMATICS, 340 (2) (2017), 51–57; IF 0.639,

[S4] A. Woryna, *On the ratio of prefix codes to all uniquely decodable codes with a given length distribution*, (under review),

[S5] A. Woryna, *On the ratio of prefix codes to all uniquely decodable codes with the three-element sequence of code-word lengths*, (under review).

In **[S1]**, we were interested in the following problem: what is the average number of temporary leaders during the round in the ski jumping competition. There are $n \geq 1$ jumpers, and after each jump a temporary leader is the jumper with the best current result. We assume that there are no favourites, and that no two jumpers will have the same score (the problem came to my mind when Adam Małysz has regularly been the winner of the competitions; in particular, the first assumption was not right). Given $1 \leq k \leq n$, we showed that the probability that there will be exactly k temporary leaders during the round is equal to $S(n,k)/n!$, where $S(n,k)$ is the corresponding Stirling number of the first kind. Consequently, we obtained that the n-th harmonic number $\sum_{i=1}^{n} 1/i$ is the solution of our problem.

In **[S2]**, we discovered new combinatorial formulae, which allow analytically to obtain a so-called optimal sequence solving a generalized version of the famous secretary problem. Previously, the solution was described only by the algorithms using mechanisms of dynamic or linear programming, which allow to compute this solution only numerically. We suppose that our purely combinatorial approach can be also used for some other (wider) classes of the secretary problem, obtaining in this way some simplifications in the currently known formulae. In the next work, we plan to use our construction of the optimal sequence to study the asymptotic behaviour of its elements, hoping to obtain answers to some still open questions concerning this asymptotics. The inspiration for writing the paper **[S2]** came after reading the book by Jakub Szczepaniak, *Matematyka nie tylko dla zakochanych*, WYDAWNICTWO POLITECHNIKI ŁÓDZKIEJ, 2010.

In **[S3]**, we investigated some important classes of variable-length codes with a given length distribution. For general variable-length codes, these sequences are well-known to be characterized by Kraft inequality, which also characterizes the length distributions of prefix codes. Most questions, even elementary, are open in this area. For example, the characterization of the length distributions of bifix codes is not known. In **[S3]**, we denoted by $UD_n(L)$ (resp. $PR_n(L)$, resp. $FD_n(L)$) the set of all codes (resp. prefix codes, resp. codes with finite delay) over an n-letter alphabet and with length distribution L. The main results of the paper are: (1) showing the inequality $|UD_n(L)|/|PR_n(L)| \geq 1+r_a r_b/|PR_n(a,b)|$, where r_a (resp. r_b) is the number of occurrences of a (resp. of b) in L, (2) showing that the equality $FD_n(L) = UD_n(L)$ holds if and only if the sequence L has length at most 2, or it is of the form (up to reordering) $L = (a, a, \ldots, a, b)$ for some a and b satisfying the divisibility $a \mid b$.

In **[S4]**, we investigated the ratio $\rho_{n,L} = |PR_n(L)|/|UD_n(L)|$ and provided some nontrivial lower and upper bounds for the numbers $\xi_{n,m} := \inf_L \rho_{n,L}$, where the infimum is taken over the sequences L of length $m \geq 1$ for which $UD_n(L) \neq \emptyset$. In particular, we obtained: $\lim_{n\to\infty} \xi_{n,m} = 1$ for every $m \geq 1$ and $\lim_{m\to\infty} \xi_{n,m} = 0$ for every $n \geq 2$. We obviously have $\xi_{n,1} = 1$ for every $n \geq 2$. In **[S4]**, we used some known characterization of codes of length two to obtain the exact value of $\xi_{n,2}$. Although the characterization of codes of length three is not known, we were able to derive the exact formula for $\xi_{n,3}$ (**[S5]**). The inspiration for exploring the subject presented in the papers **[S3], [S4], [S5]** were classes in *Information Theory and Cryptography*, which I gave to students in mathematics at the Faculty of Applied Mathematics of the Silesian University of Technology.

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