SUMMARY OF SCIENTIFIC ACHIEVEMENTS

1. **Names and surname:** Andrzej Piotr Olbryś
2. **Academic degrees:**
   (a) Master of Science in Mathematics;
   degree granted on 9th of June 2000 at University of Silesia, Katowice, Poland title of Master's Thesis: "*Theorem of Banach-Kakutani-Saks and its consequences*", supervisor: prof. dr hab. Roman Ger.
   (b) Doctor's degree in Mathematical Sciences; degree granted on 28th of June 2005 at University of Silesia, Katowice, title of Ph.D. Thesis: "*t-Wright convex functions*", supervisor: prof. dr hab. Zygfryd Kominek.
3. **Employment in academic institutions:**
   (a) February 1, 2007 - now assistant professor (full post), Functional Equations Section, Institute of Mathematics, University of Silesia.
   (b) October 1, 2005 - January 31, 2007 research and teaching assistant (part-time employment), Functional Equations Section, Institute of Mathematics, University of Silesia.
   (c) February 15, 2005 - September 30, research and teaching assistant (part-time employment), Functional Equations Section, Institute of Mathematics, University of Silesia.
4. **The scientific achievement spoken of in a respective act of Polish law**
   (a) Title of the habilitation thesis: "Separation and support theorems for selected classes of maps and its consequences".
   (b) Papers constituting the habilitation thesis:
      (I) A. Olbryś, *A support theorem for t-Wright convex functions*, Math. Inequal. Appl. 14 (2011), no. 2, 399-412.
      (II) A. Olbryś, *Representation theorems for t-Wright convexity*, J. Math. Anal. Appl. 384 (2011), no. 2, 273–283.
      (III) A. Olbryś, *On support, separation and decomposition theorems for t-Wright-concave functions*, Math. Slovaca 67 (2017), no. 3, 719–730.
      (IV) A. Olbryś, *On sandwich theorem for delta-subadditive and delta-superadditive mappings*, Results Math. 72 (2017), no. 1-2, 385–399.
      (V) A. Olbryś, *Support theorem for generalized convexity and its applications*, J. Math. Anal. Appl. 458 (2018), no. 2, 1044-1058.
      (VI) A. Olbryś, Zs. Páles, *Support theorems in abstract settings*, Publ. Math. Debrecen 93 (2018), no. 1-2, 215-240.

(c) The description of the scientific goal of the foregoing papers and results obtained, joinly with their potential applications:

The aim of the monographic set of publications is to present the separation and support theorems for important classes of maps, generalizing the classes of convex and subadditive mappings and to show their applications.

Introduction

We present here briefly the classes of functions considered in the dissertation and relations between them. For a given number $t \in (0,1)$ the symbol $\mathbb{Q}(t)$ will denote the smallest subfield of reals containing a singleton $\{t\}$. Obviously, $\mathbb{Q} \subseteq \mathbb{Q}(t)$. Let $\mathbb{K} \subseteq \mathbb{R}$ be a given field, and let X be a linear space over the field $\mathbb{K}$. A set $D \subseteq X$ is said to be *A-convex*, where $A \subseteq \mathbb{K}$, if

$$x, y \in D,\ \alpha \in A \cap [0,1] \Longrightarrow \alpha x + (1-\alpha)y \in D.$$

If $A = \{t\}$ then we say that D is a t-convex set, in the case $A = \mathbb{R}$ the set D is said to be convex.

A point $x_0 \in D$ is said to be a $\mathbb{K}$-*algebraically internal* for a set $D \subseteq X$, and we will write $x_0 \in algint_{\mathbb{K}}(D)$, if for every $x \in X$ there exists a number $\delta > 0$ such that

$$x_0 + \alpha x \in D \quad \text{for } \alpha \in (-\delta, \delta) \cap \mathbb{K}.$$

A set D is $\mathbb{K}$-*algebraically open* if $algint_{\mathbb{K}}(D) = D$. In the case, where $\mathbb{K} = \mathbb{R}$ we will write $x_0 \in algint(D)$ and the set satisfying the condition $algint(D) = D$ is said to be *algebraically open*. Any open subset of a real linear-topological space is algebraically open but the opposite implication is not true (see [62], Example 1.1).

Let D be a convex subset of a real linear space. We say that a function $f : D \to \mathbb{R}$ is *convex* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \quad \text{for } x, y \in D,\ t \in [0,1].$$

If the above inequality is satisfied for all $x, y \in D$ and fixed number $t \in (0,1)$, then we say that f is a *t-convex*. If $t = \frac{1}{2}$ then f is said to be *convex in the sense of Jensen*.

Obviously, each convex function is t-convex for all $t \in (0,1)$, in particular, convex in the sense of Jensen. The converse implication does not hold in general. Indeed, fix $t \in (0,1)$. Any discontinuous additive function $a : \mathbb{R} \to \mathbb{R}$ i.e. a solution of the Cauchy's functional equation

$$a(x+y) = a(x) + a(y), \quad x, y \in \mathbb{R},$$

satisfying additionally the condition

$$a(tx) = ta(x),\ x \in \mathbb{R}$$

is an example of t-convex and Jensen-convex function which is not convex. (proof of the existence of such function can be found, for example, in [71], Theorem 5.4.2). On the other hand, every t-convex function has to be convex in the sense of Jensen. This result was proved by N. Kuhn in [69] and the direct motivation was the paper [39] by R. Ger. An easy proof of this fact was done by Z. Daróczy and Zs. Páles in [29].

Every convex function defined on an open and convex subset of a finite-dimensional, real linear space is continuous. This is not the case for a function defined on a subset of infinite-dimensional space. Any discontinuous linear functional is an example of a discontinuous convex function. On the other hand, slight regularity conditions upon a Jensen convex function imply its continuity. The most widely known result of this type is a Bernstein-Doetsch theorem [14] (see also [62], Theorem 5.1) which states that every Jensen convex function defined on an open and convex subset of a real linear topological space which is bounded from above on some set with non-empty interior is continuous and convex. A survey of results concerning the conditions implying the continuity of additive functions and convex in the sense of Jensen can be found in M. Kuczma's monograph [71].

In [70] Kuhn investigated a more general inequality. For given, fixed numbers $s, t \in (0,1)$ a function $f : D \to \mathbb{R}$ defined on a convex set D is said to be an (s,t)-convex, if

$$f(sx + (1-s)y) \leq tf(x) + (1-t)f(y) \quad \text{for } x, y \in D.$$

In the case, where $s = t$ the class of (t,t)-convex functions coincides with the class of t-convex functions. In [70] Kuhn showed that every (s,t)-convex function is convex in the sense of Jensen. The problem of existence of non-constant solutions to the above inequality for $s \neq t$ depends on the algebraic structure of numbers s and t and was solved by J. Matkowski and M. Pycia in [77] (see also [63] for a partial solution to this problem). They proved that s and t are conjugate numbers i.e. either they are both transcendental or they are both algebraic and have the same minimal polynomial with rational coefficients if and only if there is a non-constant (s,t)-convex function.

To give a definition of convex functions in the sense of Schur we recall a few necessary notions. A relation of majorization was introduced in 1934 by G. Hardy, J. Littlewood i G. Pólya [46] in the following manner: for $x, y \in \mathbb{R}^n$

$$x \prec y \quad \Leftrightarrow \quad \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \ \text{ for } k = 1, ..., n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where, for any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $(x_{[1]}, ..., x_{[n]})$ denotes the components of x in decreasing order: $x_{[1]} \geq ... \geq x_{[n]}$. When $x \prec y$, x is said to be *majorized* by y. The relation of majorization defined above turns out to be a preordering relation i.e. it is reflexive and transitive. The fact $x \prec y$ is equivalent (see [9], [46]) to the existence of a doubly stochastic matrix [1] $S \in \mathbb{R}_n^n$ such that

$$x = Sy.$$

The functions that preserve the order of majorization (in Schur's honor who first considered them) are said to be *convex in the sense of Schur*. Thus we say that a function $f : W \to \mathbb{R}$, where $W \subseteq \mathbb{R}^n$ is *Schur convex*, if for all $x, y \in W$ the implication

$$x \prec y \Rightarrow f(x) \leq f(y)$$

holds. In the case, where $W = I^n$ with some interval $I \subseteq \mathbb{R}$ the above condition is equivalent to the following one

$$f(Sx) \leq f(x) \quad \text{for } x \in I^n$$

and for all doubly stochastic matrices $S \in \mathbb{R}_n^n$. A survey of results concerning a majorization and Schur convex functions may be found in an extensive monograph by B. C. Arnold, A. W. Marshall and I. Olkin [9].

In 1954 E.M. Wright [125] introduced a new convexity property. A function $f : D \to \mathbb{R}$ is called *Wright convex* if

$$f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y) \quad \text{for } x, y \in D,\ t \in [0,1]. \tag{1}$$

Clearly, each convex and additive function is Wright convex, and each Wright-convex functions is convex in the sense of Jensen.

The following theorem shows the connection between the classes of Schur convex and Wright convex functions:

Theorem 1 ([81], Ng, 1987) *Let $D \subseteq \mathbb{R}^m$ be a nonempty open and convex set, $f : D \to \mathbb{R}$ and $F(x_1, \ldots, x_n) = \sum_{j=1}^n f(x_j)$. The following conditions are equivalent to each other:*

(a) *F is Schur convex for some $n \geq 2$,*

(b) *F is Schur convex for every $n \geq 2$,*

(c) *f is convex in the sense of Wright,*

(d) *f admits the representation*

$$f(x) = w(x) + a(x), \quad x \in D,$$

where $w : D \to \mathbb{R}$ is convex function, and $a : \mathbb{R}^m \to \mathbb{R}$ is additive function.

[1] A doubly stochastic matrix is a square matrix $S \in \mathbb{R}_n^n$ of nonnegative real numbers, each of whose rows and columns sums to 1.

The equivalence $(c) \Leftrightarrow (d)$ gives a characterization of Wright convex functions and for functions defined on an algebraically open subset of a real linear space was proved by Kominek in [62].

If the inequality (1) is satisfied for all $x, y \in D$ and fixed number $t \in (0,1)$, then we say that f is a t-Wright convex function. The definition of t-Wright convex functions was introduced by J. Matkowski in [76]. In this paper he asked whether any t-Wright convex function has to be convex in the sense of Jensen? Gy. Maksa, K. Nikodem and Zs. Páles in [78] gave a positive answer to the problem of Matkowski for all rational $t \in (0,1)$ and for certain algebraic values of t. However, they proved that if t is either transcendental or algebraic but such that the algebraic conjugate (a root of minimal polynomial) lies outside the closed ball $\overline{B}(\frac{1}{2}, \frac{1}{2})$, then there exists a function bounded from above on the whole real line $\mathbb{R}$ which is t-Wright convex but Jensen concave and not Jensen convex. Such a function has many pathological properties, in particular, it is discontinuous at every point. In the further part of this description we will refer to this example many times.

If f is a function such that the function $-f$ is convex (t-convex, convex in the sense of Jensen, t-Wright convex), then we say that f is concave (t-concave, concave in the sense of Jensen, t-Wright concave).

If f is a function such that at the same time f and $-f$ are convex (t-convex, convex in the sense of Jensen, t-Wright convex), then f is called affine (t-affine, affine in the sense of Jensen, t-Wright affine).

The next theorem gives a characterization of t-Wright affine functions. This theorem was proved by K. Lajko [72] for functions defined on interval. The version presented here generalizes Lajko's theorem in many directions and it is a particular case of theorem proved in (O3).

Theorem 2 *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $D \subseteq X$ be a $\mathbb{Q}(t)$-convex set such that $\mathrm{algint}_{\mathbb{Q}(t)}(D) \neq \emptyset$. Then $f : D \to \mathbb{R}$ is a t-Wright affine function if and only if it has a form:*

$$f(x) = a_0 + a_1(x) + a_2(x,x), \quad x \in D,$$

where $a_0 \in \mathbb{R}$ is a constant, $a_1 : X \to \mathbb{R}$ is an additive function and $a_2 : X \times X \to \mathbb{R}$ is a bi-additive and symmetric function, satisfying the condition:

$$a_2(tx, (1-t)x) = 0 \quad \text{for } x \in X.$$

In the class of continuous functions the notions of: convexity, t-convexity, convexity in the sense of Jensen and t-Wright convexity coincide.

The definition of delta-convex maps was introduced by L. Veselý and L. Zajíček in [120] in the following manner:

Definition 1 Let X and Y be real normed spaces, and let $D \subseteq X$ be a convex set. We say that a map $F : D \to Y$ is *delta-convex*, if there exists a continuous and convex function $f : D \to \mathbb{R}$ such that $f + y^\star \circ F$ is continuous and convex for any member $y^\star \in Y^\star$ with the norm equal to 1. If this is the case, then we say that F is a delta-convex map with a control function f.

It turns out (see [120]) that a continuous map $F : D \to Y$ is delta-convex controlled by a continuous function $f : D \to \mathbb{R}$ if and only if the inequality

$$(2) \qquad \Big\| \frac{F(x)+F(y)}{2} - F\Big(\frac{x+y}{2}\Big) \Big\| \le \frac{f(x)+f(y)}{2} - f\Big(\frac{x+y}{2}\Big)$$

holds for all $x, y \in D$. Obviously, the above inequality may be investigated without any regularity assumptions upon F and f. One can easily check that in the case where $Y = \mathbb{R}$ a given map is a delta-convex if and only if it is a difference of two convex functions. The class of maps introduced by Veselý and Zajiček is a generalization (for maps taking values in vector spaces) of functions which may be representable as a differences of two convex functions.

Modifying the inequality (2) accordingly, we can consider: *delta convex maps in the sense of Jensen* [41], *delta (s,t)-convex maps* (O7), *delta subadditive maps* [40], (IV), *delta Schur convex maps* (O8) e.t.c. as a natural generalization of a class of functions which are a difference of two convex functions in the sense of Jensen, (s,t)-convex functions, subadditive functions and Schur convex functions.

The support and separation theorems strictly related to a classical Hahn-Banach theorem have applications in many branches of modern functional analysis, convex geometry, optimization theory and economics. The support theorems lead to the representation of convex functions as the pointwise maximum of affine functions, subadditive functions as the pointwise maximum of additive functions and convex sets as the intersection of half spaces. From the appropriate version of the support theorem we can deduce the nonemptyness of the subgradient of a convex function at a given point and the Fenchel-Moreau duality theorem which has many applications in optimization theory, financial mathematics and economics.

A main tool to prove the support theorems is to use the Hahn-Banach extension theorem or sandwich theorem or one of their generalizations ([8], [11], [12], [18], [19], [22], [35]-[38], [58], [67], [68], [79], [88]-[93], [105], [106], [111], [115], [116]). A survey of results concerning the Hahn-Banach theorem is contained in [7]. A classical separation theorem of Kakutani [57] says that two disjoint convex sets in a real linear space can be separated by a halfspace i.e. a convex set with the convex complement. This theorem is known as a geometric version of the Hahn-Banach theorem. The separation problem was intensively studied by many mathematicians. A classical result is the S. Mazur and W. Orlicz theorem [79], which was later generalized by R. Kaufman [58] and then further developed by P. Kranz [68]. In 1978 roku G. Rodé [105] proved an abstract version of the Hahn-Banach theorem to setting of convexity defined in terms of families of commuting operations. To present this theorem we introduce a few necessary notions.

Let X be a non-empty set, $m \in \mathbb{N}$. Denote by $\mathcal{P}^m(X)$ the family of all pairs (σ, s) such that $\sigma : X^m \to X$ is an arbitrary function and there exist $s_0 \in \mathbb{R}$ and $s_1, \ldots, s_m \in [0, \infty)$ such that $s : \mathbb{R}^m \to \mathbb{R}$ is an affine function of the form:

$$s(y_1, \ldots, y_m) := s_0 + s_1 y_1 + \ldots + s_m y_m.$$

Let $\mathcal{P}(X) := \cup_{m \in \mathbb{N}} \mathcal{P}^m(X)$ and let $\Pi \subseteq \mathcal{P}(X)$ be a fixed subset. Put $\Pi^m := \Pi \cap \mathcal{P}^m(X),\ m \in \mathbb{N}$. The set Π is *commutative* if for any $m, n \in \mathbb{N}$, the operations $(\sigma, s) \in \Pi^m, (\tau, t) \in \Pi^n$ commute i.e.

$$\sigma(\tau(x_1^1, \ldots, x_n^1), \ldots, \tau(x_1^m, \ldots, x_n^m)) = \tau(\sigma(x_1^1, \ldots, x_1^m), \ldots, \sigma(x_n^1, \ldots, x_n^m))$$

for all $x_i^j \in X,\ i = 1, \ldots, n,\ j = 1, \ldots, m$ and

$$s(t(y_1^1, \ldots, y_n^1), \ldots, t(y_1^m, \ldots, y_n^m)) = t(s(y_1^1, \ldots, y_1^m), \ldots, s(y_n^1, \ldots, y_n^m))$$

for all $y_i^j \in \mathbb{R}$, $i = 1, \ldots, n,\ j = 1, \ldots, m$.

A function $f : X \to [-\infty, \infty)$ is called Π-*convex* if

$$f(\sigma(x_1, \ldots, x_m)) \leq s_0 + s_1 f(x_1) + \ldots + s_m f(x_m),$$

for all $m \in \mathbb{N},\ (\sigma, s) \in \Pi^m$ and $x_1, \ldots, x_m \in X$. f is said to be a Π-*concave* if the function $-f$ is Π-convex. If f is at the same time Π-convex and Π-concave then we say that it is a Π-*affine function.*

Theorem 3 ([105], G. Rodé, 1978) *Let $\Pi \subseteq \mathcal{P}(X)$ be a commutative family of operations and let $f : X \to \mathbb{R}$ be a Π-convex function. Let*

$$M(\Pi, f) := \{g : X \to [-\infty, \infty) \mid g \text{ is } \Pi\text{-concave and } g \leq f\}.$$

Then any maximal element of $M(\Pi, f)$ is a Π-affine function.

The above result and its generalization (see P. Volkmann, H. Weigel [122]) is one of the most general versions of the Hahn-Banach theorem and its simple proof can be found in the paper [67] by König. The geometric version of Róde theorem was proved by Páles [93] whereas the necessary and sufficient conditions for the separation by Π-convex, Π-concave and Π-affine functions can be found in the paper [88].

The support and separation problems for t-Wright convex functions

The papers (I)-(III) are devoted to the problem of support and separation for t-Wright convex and t-Wright concave functions and the applications of the obtained results to the characterizations of some subclasses of the class of t-Wright convex functions. In the paper (I) we deal with a problem of separation for t-Wright convex and t-Wright concave functions and we prove a support theorem.

A separation theorem for t-convex functions, so in particular, also for convex functions in the sense of Jensen, is a consequence of Rodé's theorem. Unfortunately, Rodé's theorem does not apply to the t-Wright convex functions. The following theorem generalizes the analogical theorem for convex functions in the sense of Jensen ($t = \frac{1}{2}$):

Theorem 4 ((I), Theorem 2) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $D \subseteq X$ be a t-convex set. If $f : D \to \mathbb{R}$ is a t-Wright convex function, $g : D \to \mathbb{R}$ is a t-Wright concave function and*

$$g(x) \le f(x), \quad x \in D,$$

then there exists a t-Wright affine function $a : D \to \mathbb{R}$ satisfying

$$g(x) \le a(x) \le f(x), \quad x \in D.$$

Recall, that a function $a_y : D \to \mathbb{R}$ is called a support function of the function $f : D \to \mathbb{R}$ at the point $y \in D$ if it fulfills conditions

(i) $a_y(y) = f(y)$,

(ii) $a_y(x) \le f(x)$ for $x \in D$.

It is known that every t-convex function (convex function in the sense of Jensen) can be supported at any algebraically internal point of the domain by a t-affine function (affine function in the sense of Jensen). This theorem is also a consequence of Rodé's result, and alternative proofs of it can be found in the papers [62], [63], [70], [86]. In these proofs the key fact is used, that every function convex in the sense of Jensen defined on an algebraically open and convex set and taking values in the set $[-\infty, \infty)$ is either identically equal to $-\infty$ or takes only real values. It is easy to observe that for $t \in (0,1) \setminus \{\frac{1}{2}\}$ the function $f : \mathbb{R} \to [-\infty, \infty)$ given by formula

$$f(x) = \begin{cases} 0 & , \ x = x_0 \\ -\infty & , \ x \neq x_0 \end{cases}$$

is t-Wright convex. The proof of a support theorem in this case required the use of a new method. The theorem that we present below is the key tool. To present this theorem we introduce a few necessary notions.

For a given point $y \in D$ let $D_y := (2y - D) \cap D$. It is a maximal set symmetric with respect to y contained in D (i.e. $D_y = 2y - D_y$). The set D_y inherits convexity from the set D. Next, we define recursively the sequences of means $M_n, N_n : D \times D \to D$

$$M_1(x,z) = M(x,z) := tx + (1-t)z, \qquad N_1(x,z) = N(x,z) := (1-t)x + tz,$$

and next

$$M_{n+1}(x,z) := M(M_n(x,z), N_n(x,z)), \qquad N_{n+1}(x,z) := N(M_n(x,z), N_n(x,z)).$$

Obviously, $M_n(x,z) = N_n(z,x)$, $M_n(x,x) = N_n(x,x) = x$, $x, z \in D$, $n \in \mathbb{N}$. Moreover, if $f : D \to \mathbb{R}$ is a t-Wright convex function, then for all $x, z \in D$ and $n \in \mathbb{N}$ we have

$$f(M_{n+1}(x,z)) + f(N_{n+1}(x,z)) \leq f(M_n(x,z)) + f(N_n(x,z)).$$

Therefore, for all $x, z \in D$, there exists the limit:

$$\lim_{n\to\infty} [f(M_n(x,z)) + f(N_n(x,z))].$$

The finiteness of the above limit plays a key role in further considerations.

Theorem 5 ((I), Theorem 3) *Let X be a linear space over the field $\mathbb{K}$ where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $D \subseteq X$ be a t-convex set. Assume that $f : D \to \mathbb{R}$ is a t-Wright convex function. If $y \in D$ and*

$$(\star) \qquad \lim_{n\to\infty} [f(M_n(x, 2y-x)) + f(N_n(x, 2y-x))] > -\infty \quad \textit{for } x \in D_y,$$

then the function $A_y : D_y \to \mathbb{R}$ given by the formula

$$(3) \qquad A_y(x) = \lim_{n\to\infty} [f(M_n(x, 2y-x)) + f(N_n(x, 2y-x))], \quad x \in D_y$$

is t-Wright affine. If, moreover, $y \in algint_{\mathbb{Q}(t)}(D)$, then the formula (3) *defines a t-Wright affine function $A_y : X \to \mathbb{R}$.*

The proof of the previous theorem was preceded by several technical lemmas which allow us to state that the function A_y given by formula (3) is t-Wright affine.

Observe that, if f is a convex function in the sense of Jensen ($t = \frac{1}{2}$) then for an arbitrary $x \in D_y$ we have

$$A_y(x) = 2f(y), \quad x \in D_y.$$

In the paper (II, Observation 1) we showed more. Namely, if $y \in algint_{\mathbb{Q}(t)}(D)$ then the function $A_y : X \to \mathbb{R}$ given by formula (3) has the form

$$A_y(x) = 2f(y) + a_2(y-x, y-x), \quad x \in X,$$

where $a_2 : X \times X \to \mathbb{R}$ is a bi-additive and symmetric function. Moreover, since a_2 is a function obtained by applying the generalized version of the Lajko's theorem (Theorem 2) so in addition

$$a_2(tx, (1-t)x) = 0 \quad \text{for } x \in X.$$

Using the separation theorem (Theorem 4) and Theorem 5 we get the following support theorem

Theorem 6 ((I), Theorem 4) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $D \subseteq X$ be a t-convex set. Assume that $f : D \to \mathbb{R}$ is a t-Wright convex function and $y \in D$. Then there is a t-Wright affine support function of $f_{|D_y}$ at y i.e. the function $a_y : D_y \to \mathbb{R}$ satisfying*

(i) $a_y(y) = f(y)$,

(ii) $a_y(x) \leq f(x) \quad$ *for* $x \in D_y$,

(iii) $a_y(tx + (1-t)z) + a_y((1-t)x + tz) = a_y(x) + a_y(z) \quad$ *for* $x, z \in D_y$

if and only if

$$(\star) \qquad \lim_{n\to\infty} [f(M_n(x, 2y-x)) + f(N_n(x, 2y-x))] > -\infty \quad \text{for } x \in D_y.$$

Moreover, if $y \in algint_{\mathbb{Q}(t)}(D)$ and the condition $(\star)$ holds, then there is a t-Wright affine support function $a_y : D \to \mathbb{R}$ of f at y defined on the whole domain D.

Remark 1 *An example* [74, Example] *shows that a t-Wright convex function does not need to satisfy the condition $(\star)$.*

It follows from the proof of the above theorem that the support function can be estimated from above and from below in the following manner

$$A_y(x) - f(2y-x) \leq a_y(x) \leq f(x), \quad x \in D_y.$$

In the case where f is a function, convex in the sense of Jensen we get the estimation

$$2f(y) - f(2y-x) \leq a_y(x) \leq f(x), \quad x \in D_y.$$

The above estimation is very useful, in particular, if the function f is bounded on the set D_y, then the support function a_y inherits this property.

The next theorem gives the conditions equivalent to the existence of a support at a given point for functions defined on $\mathbb{Q}(t)$-algebraically open sets. It turns out that the existence of a support function at one point guarantees its existence at any point.

Theorem 7 ((I), Theorem 5) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $D \subseteq X$ be a $\mathbb{Q}(t)$-algebraically open and t-convex set. Assume that $f : D \to \mathbb{R}$ is a t-Wright convex function. Then the following conditions are equivalent to each other:*

(i) *there exists the point $y \in D$ such that*

$$(\star) \qquad \lim_{n\to\infty} [f(M_n(x, 2y - x)) + f(N_n(x, 2y - x))] > -\infty, \quad x \in D_y;$$

(ii) *there exists t-Wright concave function $g : D \to \mathbb{R}$ such that*

$$g(x) \le f(x), \quad x \in D;$$

(iii) *for an arbitrary point $y \in D$ the condition $(\star)$ holds true;*

(iv) *for an arbitrary point $y \in D$ there exists a t-Wright affine support of f.*

Support and separation theorems have many consequences. In the paper (II) we give the applications of the proved theorems to the characterization of some subclasses of the class of all t-Wright convex functions. In the paper [66] Maksa, Nikodem and Páles gave the algebraic conditions (dependent on the algebraic structure of the number $t \in (0, 1)$) which imply that any t-Wright convex function is convex in the sense of Jensen. The following theorem gives the topological necessary and sufficient conditions under which any t-Wright convex function is convex in the sense of Jensen.

Theorem 8 ((II), Theorem 5) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $D \subseteq X$ be a $\mathbb{Q}(t)$-algebraically open and $\mathbb{Q}(t)$-convex. Assume that $f : D \to \mathbb{R}$ is a t-Wright convex function. Then f is convex in the sense of Jensen if and only if for an arbitrary point $y \in D$ there exists the set $U_y \subseteq D$ symmetric with respect to y, such that $y \in \mathrm{algint}_{\mathbb{Q}(t)}(U_y)$ and the function*

$$U_y \ni x \longrightarrow f(x) + f(2y - x)$$

is bounded from below.

Clearly, any function which is locally bounded from below satisfies the above condition. In locally-convex linear topological space a more general theorem holds true:

Theorem 9 ((II), Theorem 9) *Let D be an open and convex subset of a real locally convex linear-topological space, and let $f : D \to \mathbb{R}$ be a t-Wright convex function. If f is bounded from below on a neighborhood of some point $x_0 \in D$ then it is convex in the sense of Jensen.*

The next theorem and corollary give further necessary and sufficient conditions on the convexity in the sense of Jensen of a t-Wright convex function.

Theorem 10 ((II), Theorem 6) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $D \subseteq X$ be a $\mathbb{Q}(t)$-algebraically open and $\mathbb{Q}(t)$-convex set. Assume that $f : D \to \mathbb{R}$ is a t-Wright convex function. Then f is a convex function in the sense of Jensen if and only if there exists a function $\Phi : D \to \mathbb{R}$ such that*

$$\Phi\Big(\frac{x+y}{2}\Big) \leq f(x) + f(y) \quad \textit{for } x, y \in D.$$

Corollary 1 ((II), Corollary 1) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $D \subseteq X$ be a $\mathbb{Q}(t)$-algebraically open and $\mathbb{Q}(t)$-convex set. Assume that $f : D \to \mathbb{R}$ is a t-Wright convex function. Then f is a convex function in the sense of Jensen if and only if there exists a convex function in the sense of Jensen $g : D \to \mathbb{R}$ such that*

$$g(x) \leq f(x), \quad x \in D.$$

It turns out that the function f in Theorems 9, 10 and Corollary 1 does not need be a t-convex. In the paper (II) we give a suitable example.

The next theorem states that the condition $(\star)$ in fact characterizes certain important subclass of the class of t-Wright convex functions.

Theorem 11 ((II), Theorem 10) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $D \subseteq X$ be a $\mathbb{Q}(t)$-algebraically open and $\mathbb{Q}(t)$-convex set. Assume that $f : D \to \mathbb{R}$ is a t-Wright convex function. Then there exists the point $y \in D$ such that*

$$(\star) \qquad \lim_{n\to\infty} [f(M_n(x, 2y-x)) + f(N_n(x, 2y-x))] > -\infty \quad \textit{for all } x \in D_y$$

if and only if

$$f(x) = a(x) + g(x), \quad x \in D,$$

where $a : D \to \mathbb{R}$ is a t-Wright affine function and $g : D \to \mathbb{R}$ is a convex function in the sense of Jensen.

Using the separation theorem (Theorem 4) we obtain the following majorization result:

Theorem 12 ((II), Theorem 11) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $D \subseteq X$ be a $\mathbb{Q}(t)$-algebraically open and $\mathbb{Q}(t)$-convex set. Assume that $f : D \to \mathbb{R}$ is a t-Wright convex function and $g : D \to \mathbb{R}$ is a t-Wright concave function. If*

$$g(x) \leq f(x), \quad \textit{for } x \in D,$$

then there exist: a convex function in the sense of Jensen $F : D \to \mathbb{R}$, a concave function in the sense of Jensen $G : D \to \mathbb{R}$ and a t-Wright affine function $a : D \to \mathbb{R}$ such that

$$f(x) = a(x) + F(x), \qquad g(x) = a(x) + G(x), \quad x \in D.$$

The next theorem from the paper (II) establishes the necessary and sufficient conditions under which a t-Wright convex function is a Wright convex function.

Theorem 13 ((II), Theorem 12) *Let D be an algebraically open and convex subset of a real linear space X and let $f : D \to \mathbb{R}$ be a t-Wright convex function. Then the following conditions are pairwise equivalent:*

(i) *f is a Wright convex function;*

(ii) *there exist: an additive function $a : X \to \mathbb{R}$ and convex function $w : D \to \mathbb{R}$ such that*

$$f(x) = a(x) + w(x), \quad x \in D;$$

(iii) *for arbitrary $x, y \in D$:*

$$\lim_{\alpha \to \frac{1}{2}} [f(\alpha x + (1-\alpha)y) + f((1-\alpha)x + \alpha y)] = 2f\Big(\frac{x+y}{2}\Big);$$

(iv) *there exists the function $\Phi : D \to [0, \infty)$ such that*

$$\bigwedge_{x,y \in D} \bigvee_{\lambda_{xy} \in (0, \frac{1}{2})} \bigwedge_{\lambda \in (\frac{1}{2} - \lambda_{xy}, \frac{1}{2} + \lambda_{xy})} |f(\lambda x + (1-\lambda)y) + f((1-\lambda)x + \lambda y)| \le \Phi\Big(\frac{x+y}{2}\Big).$$

The above theorem provides further arguments to consideration the introduced by Kominek [66] subclass W_0 of the class of t-Wright convex functions defined as follows:

$$W_0 := \{f : D \to \mathbb{R} : f = a + g, \text{ where } a : X \to \mathbb{R} \text{ is an additive function, } g : D \to \mathbb{R} \text{ is continuous and convex}\}.$$

Obviously, if D is an open and convex subset of a finite-dimensional space, then the class W_0 coincides with the class of Wright convex functions, but in infinite-dimensional spaces this class is a substantially smaller.

Theorem 14 ((II), Theorem 13) *Let D be an algebraically open and convex subset of a real linear space X and let $f : D \to \mathbb{R}$ be a t-Wright convex function. Then the following conditions are pairwise equivalent*

(i) *$f \in W_0$;*

(ii) *there exists the point* $y \in D$ *such that*

$$\lim_{x \to y}[f(x) + f(2y - x)] = 2f(y);$$

(iii) *there exists the point* $y \in D$ *and symmetric with respect to* y *its neighborhood* [2] $U_y \subseteq X$ *such that the function*

$$U_y \cap D \ni x \longrightarrow f(x) + f(2y - x)$$

is bounded.

In the paper (III) we consider analogical problems as in (I) and (II) for t-Wright convex functions defined on the whole space. The assumption that the domain of the considered functions is the whole space is important, and in most theorems can not be omitted. Let us observe that if $t \neq \frac{1}{2}$ then the function $f : X \to \mathbb{R}$ is a t-Wright convex if and only if

$$f(x) + f(y) \leq f\Big(\frac{t}{2t-1}x + \frac{1-t}{2t-1}y\Big) + f\Big(\frac{1-t}{2t-1}x + \frac{t}{2t-1}y\Big) \quad \text{for } x, y \in X.$$

In the whole paper (III) we assume that $t \neq \frac{1}{2}$. In spite of using there the analogous method as in the paper (I) and (II) the results contained therein are so surprising and unexpected that it is worth to quoting them here. Let us start with the following separation theorem:

Theorem 15 ((III), Theorem 2) *Let* X *be a linear space over the field* $\mathbb{K}$, *where* $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$, *let* $f : X \to \mathbb{R}$ *be a* t*-Wright convex function, and let* $g : X \to \mathbb{R}$ *be a* t*-Wright-concave function. If*

$$f(x) \leq g(x), \quad x \in X,$$

then there exists a t*-Wright affine function* $a : X \to \mathbb{R}$ *such that*

$$f(x) \leq a(x) \leq g(x) \quad \textit{for } x \in X.$$

It can be shown (using e.g. Theorem 7.1 from [62]) that in the case where f and $-g$ are convex functions in the sense of Jensen, defined on the whole linear space (over the field of real numbers) then the inequality $f \leq g$ implies that f and g are affine functions in the sense of Jensen differing by a constant.

[2] A printing mistake has crept into the paper (II), there is in it an assumption that $y \in algint_{\mathbb{R}}(U_y)$, although the proof was carried out for the set U_y which is open in the space X.

Let us define recursively the sequences of maps $K_n, L_n : X \times X \to X$ by

$$K_1(x,z) = K(x,z) = \frac{t}{2t-1}x + \frac{t-1}{2t-1}z, \qquad L_1(x,z) = L(x,z) = \frac{t-1}{2t-1}x + \frac{t}{2t-1}z,$$

and

$$K_{n+1}(x,z) := K(K_n(x,z), L_n(x,z)), \qquad L_{n+1}(x,z) := L(K_n(x,z), L_n(x,z)), \text{ for } n \in \mathbb{N}.$$

Obviously, $K_n(x,z) = L_n(z,x),\ K_n(x,x) = L_n(x,x) = x,\ x,z \in X,\ n \in \mathbb{N}$. Moreover, if $f : X \to \mathbb{R}$ is a t-Wright convex function, then for all $x,z \in X$ and $n \in \mathbb{N}$ we have

$$f(K_n(x,z)) + f(L_n(x,z)) \le f(K_{n+1}(x,z)) + f(L_{n+1}(x,z)).$$

This time we get the existence of the limit:

$$\lim_{n\to\infty} [f(K_n(x,z)) + f(L_n(x,z))],$$

for all $x,z \in X$, and the following theorem can be interpreted as a support theorem for t-Wright concave functions:

Theorem 16 ((III), Theorem 5) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a t-Wright convex function ($t \ne \frac{1}{2}$), $y \in X$. Then there exists a t-Wright affine function $a_y : X \to \mathbb{R}$ such that*

$$a_y(y) = f(y) \qquad \text{and} \qquad f(x) \le a_y(x),\ x \in X$$

if and only if

$$(\diamond) \qquad \lim_{n\to\infty} [f(K_n(x,2y-x)) + f(L_n(x,2y-x))] < \infty, \quad x \in X.$$

In the case where f is a convex function in the sense of Jensen, the existence of an affine function in the sense of Jensen $a_y : X \to \mathbb{R}$ fulfilling $f(y) = a_y(y),\ f(x) \le a_y(x),\ x \in X$ is possible only in the case where f is an affine function in the sense of Jensen but then this theorem is trivial. However, there are t-Wright convex functions fulfilling condition $(\diamond)$ which are not t-Wright affine. This property, for example, has the mentioned function constructed by Maksa, Nikodem and Páles in the paper [74]. Clearly, each such a function has to be discontinuous at every point.

Two conditions equivalent to the condition $(\diamond)$ gives the following theorem

Theorem 17 ((III), Theorem 6) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a t-Wright convex functions. Then the following conditions are pairwise equivalent*

(a) *there exists a point $y \in X$ such that*

$$\lim_{n\to\infty} [f(K_n(x,2y-x)) + f(L_n(x,2y-x))] < \infty \quad \text{for } x \in X;$$

(b) *there exists a t-Wright concave function $g : X \to \mathbb{R}$, such that*

$$f(x) \le g(x) \quad \text{for } x \in X;$$

(c) *for an arbitrary $y \in X$ we have*

$$\lim_{n\to\infty} [f(K_n(x, 2y - x)) + f(L_n(x, 2y - x))] < \infty \quad \text{for } x \in X.$$

The next theorem gives the necessary and sufficient conditions under which any t-Wright convex function is convex in the sense of Jensen. The example given by Maksa, Nikodem and Páles shows that such functions exist and have pathological properties.

Theorem 18 ((III), Theorem 7) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a t-Wright convex function. Then f is a convex function in the sense of Jensen if and only if there exists a function $\Psi : X \to \mathbb{R}$ such that*

$$f(x) + f(y) \le \Psi\Big(\frac{x+y}{2}\Big), \quad x, y \in X.$$

As a consequence of the above theorem and Theorem 9 we get the necessary and sufficient conditions under which any t-Wright convex function is affine in the sense of Jensen.

Theorem 19 ((III), Theorem 8) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a t-Wright convex function. Then f is an affine function in the sense of Jensen if and only if there exists a function $G : X \to \mathbb{R}$ such that*

$$|f(x) + f(y)| \le G\Big(\frac{x+y}{2}\Big), \quad x, y \in X.$$

In particular, the following theorem follows from Theorem 18.

Theorem 20 ((III), Theorem 9) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a t-Wright convex function. If f is bounded from above then it is concave in the sense of Jensen.*

Therefore, we see that the concavity in the sense of Jensen of a discontinuous t-Wright convex function, bounded from above on the whole real line in the example given by Maksa, Nikodem and Páles, although it surprises, is a normal behavior. In view of the above theorem it was enough to provide any discontinuous t-Wright convex function, bounded from above on the whole real line.

I follows from the proof of the above theorem that it is enough to postulate, the boundedness from above of the function f, intuitively "far away". In the case where

X is a linear-topological space it is enough to assume that the function f is bounded from above on some proper neighborhood of zero.

It turns out that the condition ($\diamond$) characterizes some subclass of the class of t-Wright convex functions defined on the whole space. The following theorem speaks about it.

Theorem 21 *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a t-Wright convex function. Then f satisfies the condition ($\diamond$), i.e.*

$$\lim_{n\to\infty} [f(K_n(x, 2y - x)) + f(L_n(x, 2y - x))] < \infty, \quad x \in X,$$

if and only if f has the form

$$f(x) = a(x) + g(x), \quad x \in X,$$

where $a : X \to \mathbb{R}$ is a t-Wright affine function and $g : X \to \mathbb{R}$ is a concave function in the sense of Jensen.

The last characterization concerns the t-Wright convex functions which are majorized by t-Wright concave ones.

Theorem 22 *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$. Assume that $f : X \to \mathbb{R}$ is a t-Wright convex function and $g : X \to \mathbb{R}$ is a t-Wright concave function. If*

$$f(x) \le g(x) \quad \text{for } x \in X,$$

then there exist: concave functions in the sense of Jensen $F, -G : X \to \mathbb{R}$ and a t-Wright affine function $a : X \to \mathbb{R}$ such that

$$f(x) = a(x) + F(x), \qquad g(x) = a(x) + G(x), \quad x \in X.$$

The equivalent of the Kranz theorem for delta-subadditive mappings

In the paper (IV) we consider the problems of separation and support for delta-subadditive and delta superadditive maps. Ger in [40] was the first who considered the delta-subadditive maps. The discussed paper (IV) is a partial generalization of results obtained by Kranz [68], Z. Gajda and Kominek [37].

Assume that $(Y, \|\cdot\|)$ is a real Banach space and $(S, \cdot)$ is a weakly commutative semigroup i.e. a semigroup satisfying the condition

$$\bigwedge_{x,y \in S} \bigvee_{n \in \mathbb{N}} (x \cdot y)^{2^n} = x^{2^n} \cdot y^{2^n}.$$

The powers of the form x^{2^n} are defined recursively: $x^{2^0} = x,\ x^{2^{k+1}} = x^{2^k} \cdot x^{2^k}$. The concept of weakly commutative semigroup was introduced by Józef Tabor in [114]. It is clear that every commutative group is weakly commutative but there exist non-commutative semigroups and even groups which are weekly commutative.

Definition 2 A map $F : S \to Y$ is called *delta-subadditive* with a control function $f : S \to \mathrm{I\!R}$ (we will write shortly $(F, f) \in D_s(S)$), if the inequality

$$\|F(x) + F(y) - F(x \cdot y)\| \le f(x) + f(y) - f(x \cdot y)$$

holds for every $x, y \in S$. In the case where $(-F, -f) \in D_s(S)$ we say that F is a *delta-superadditive* with a control function f.

It is easy to check that if $(F, f), (-F, -f) \in D_s(S)$ then the mappings f and F have to be additive.

The classical theorem on separation obtained by Kranz has the following form:

Theorem 23 ([68], Kranz, 1972) *Let $(S, \cdot)$ be a commutative semigroup. Assume that $f : S \to \mathrm{I\!R}$ is a subadditive function, i.e.*

$$f(x \cdot y) \le f(x) + f(y) \quad \textit{for } x, y \in S,$$

$g : S \to \mathrm{I\!R}$ is a superadditive function, i.e.

$$g(x \cdot y) \ge g(x) + g(y) \quad \textit{for } x, y \in S$$

and

$$g(x) \le f(x) \quad \textit{for } x \in S.$$

Then there exists an additive function $a : S \to \mathrm{I\!R}$ such that

$$g(x) \le a(x) \le f(x) \quad \textit{for } x \in S.$$

Let $(Y, \|\cdot\|)$ be a real normed space. Consider the linear space $\overline{Y} := Y \times \mathbb{R}$, where the addition and scalar multiplication are defined coordinatewise. Let us recall that for a given positive number ε the convex cone defined by formula

$$\mathcal{K}_\varepsilon := \{(x,t) \in \overline{Y} : \ \varepsilon\|x\| \le t\}$$

is called the *Lorentz* cone. This cone induces in $\overline{Y}$ a partial order in the following manner:

$$(x_1, t_1) \preceq_{\mathcal{K}_\varepsilon} (x_2, t_2) \Leftrightarrow \varepsilon\|x_2 - x_1\| \le t_2 - t_1.$$

This partial order is compatible with the linear structure of $\overline{Y}$, i.e.

- $x \preceq_{\mathcal{K}_\varepsilon} y \Rightarrow x + z \preceq_{\mathcal{K}_\varepsilon} y + z \quad$ for $x, y, z \in \overline{Y}$,
- $x \preceq_{\mathcal{K}_\varepsilon} y \Rightarrow \alpha x \preceq_{\mathcal{K}_\varepsilon} \alpha y \quad$ for $x, y \in \overline{Y}, \ \alpha \ge 0$.

Note that, defining for given maps $F : S \to Y$ and $f : S \to \mathbb{R}$ the map $\overline{F} : S \to \overline{Y}$ via the formula

$$\overline{F}(x) := (F(x), f(x)), \quad x \in S,$$

we can rewrite the inequality defining the notion of delta-subadditivity of the map F with a control function f by the formula

$$\overline{F}(x \cdot y) \preceq_{\mathcal{K}_1} \overline{F}(x) + \overline{F}(y), \quad x, y \in S,$$

where $\mathcal{K}_1 = \{(x,t) \in \overline{Y} : \|x\| \le t\}$. This remark shows that delta-subadditive maps generalize subadditive maps by replacing the classical inequality by the relation of partial order induced by the Lorentz cone. Classic results for subadditive functions are obtained by putting $F = 0$. For delta-subadditive maps we have the following version of the separation theorem

Theorem 24 ((IV), Theorem 1) *Let $(S, \cdot)$ be a commutative semigroup, and let $(Y, \|\cdot\|)$ be a real Banach space. Assume that $F : S \to Y$ is a delta-subadditive map with a control function $f : S \to \mathbb{R}$, and $G : S \to Y$ is a delta-superadditive map with a control function $g : S \to \mathbb{R}$. Suppose that $(G, g) \preceq_{\mathcal{K}_1} (F, f)$, i.e.*

$$\|F(x) - G(x)\| \le f(x) - g(x), \quad x \in S.$$

If, moreover,

$$\sup\{f(x) - g(x) : x \in S\} < \infty,$$

then there exist unique additive mappings $A : S \to Y$ and $a : S \to \mathbb{R}$ such that

$$(G(x), g(x)) \preceq_{\mathcal{K}_1} (A(x), a(x)) \preceq_{\mathcal{K}_1} (F(x), f(x)), \quad x \in S.$$

The above theorem generalizes the Theorem 1 from the paper [37] by Gajda and Kominek. As an application we obtain an easy proof of the classical Hyers-Ulam stability result for the Cauchy's equation.

Theorem 25 ((IV), Corollary 2) *Let $(S, \cdot)$ be a weekly commutative semigroup and let $(Y, \|\cdot\|)$ be a real Banach space. If $F : S \to Y$ is an ε-additive map, i.e.*

$$\|F(x) + F(y) - F(x \cdot y)\| \leq \varepsilon \quad \textit{for } x \in S,$$

where $\varepsilon > 0$, then there exists a unique additive map $A : S \to Y$ such that

$$\|F(x) - A(x)\| \leq \varepsilon \quad \textit{for } x \in S.$$

The stability problem was formulated by S. Ulam in 1940 who asked whether if we distort the Cauchy's functional equation, i.e. we will postulate its fulfillment by a given function with some accuracy is there a solution to the Cauchy's equation uniformly close to this function? A positive solution to the Ulam's problem, for functions mapping one Banach space into another, was given first by D. H. Hyers [50].

We obtained the necessary and sufficient conditions for the existence of a support function at a given point for maps defined on abelian groups uniquely divisible by 2. An abelian group $(G, +)$ is said to be a *uniquely* 2*-divisible* if for an arbitrary $x \in G$ there exists a unique element $y \in G$ such that $2y = x$; this element is denoted by $\frac{1}{2}x$. Using the support theorem proved in the paper (O7, part 5, p. 29) we get the following theorem.

Theorem 26 ((IV), Theorem 3) *Let $(X, +)$ be a uniquely divisible by 2 abelian group, and let $(Y, \|\cdot\|)$ be a real Banach space. Let $F : X \to Y$ be a delta-subadditive map with a control function $f : X \to \mathbb{R}$ and let $y \in X$. Then there exist additive maps $A_y : X \to Y$, $a_y : X \to \mathbb{R}$ such that*

$$\|F(x) - A_y(x)\| \leq f(x) - a_y(x) \quad \textit{for } x \in X,$$

and

$$A_y(y) = F(y), \qquad a_y(y) = f(y),$$

if and only if

$$f(y^n) = nf(y) \quad \textit{for } n \in \mathbb{N}.$$

A support theorem for sublinear functions was proved by Berz in the paper [15]. An analogous result for real-valued subadditive and $\mathbb{N}$-homogeneous functions defined on weakly commutative semigroup was done by Gajda and Kominek [37].

A support theorem for generalized convexity

In the literature one can find many generalizations of the notion of t-convexity, convexity in the sense of Jensen and convexity, which consist in distorting the right hand side of the inequality with the left hand side unchanged (e.g. the concept of an approximate-convexity, strong-convexity, delta-convexity, convexity in the sense of Wright and many others). In the paper (V) the definition of the class of functions that includes all classes of this type has been proposed.

Let D be a t-convex (convex) subset of a real linear space, and let $w : D \times D \times [0,1] \to \mathbb{R}$ be a given function. For a given number $t \in [0,1]$, a function $f : D \to \mathbb{R}$ is said to be

(ω, t)-*convex*, if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \omega(x,y,t) \quad \text{for } x, y \in D; \tag{4}$$

(ω, t)-*concave*, if

$$tf(x) + (1-t)f(y) + \omega(x,y,t) \le f(tx + (1-t)y) \quad \text{for } x, y \in D;$$

(ω, t)-*affine*, if

$$tf(x) + (1-t)f(y) + \omega(x,y,t) = f(tx + (1-t)y) \quad \text{for } x, y \in D.$$

If the above inequalities are satisfied for all numbers $t \in [0,1]$ then we say that f is ω-*convex* (ω-*concave*, ω-*affine*, respectively). The notion of ω-convexity is a common generalization of the notion of usual convexity, strong-convexity, approximate-convexity, delta-convexity and many others. These classes of functions have been extensively studied by many mathematicians, see for example: [10], [51], [96], [97], [107], [113], [120], [125]. We have

- $\omega = 0$: for convexity,
- $\omega(x,y,t) = -ct(1-t)\|x-y\|^2, \ c > 0$: for strong convexity,
- $\omega(x,y,t) = c\|x-y\|^\gamma, \ c > 0, \gamma \ge 0$: for approximate-convexity,
- $\omega(x,y,t) = -\|tF(x) + (1-t)F(y) - F(tx + (1-t)y)\|$: for delta-convexity,
- $\omega(x,y,t) = (1-t)f(x) + tf(y) - f((1-t)x + ty)$: for Wright-convexity.

Note that, without any additional assumptions on ω nothing can be said about the function f. Indeed, an arbitrary function $f : D \to \mathbb{R}$ is an (ω, t)-affine, so in particular (ω, t)-convex and (ω, t)-concave with the function

$$\omega(x,y,t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y).$$

The main result of the paper (V) is the theorem providing the necessary and sufficient conditions on ω for the existence of an (ω, t)-affine support function at a point.

Theorem 27 ((V), Theorem 3) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$, let D be a t-convex set and let $y \in algint_{\mathbb{Q}(t)}(D)$. Assume, that $f : D \to \mathbb{R}$ is an (ω, t)-convex function, where $\omega : D \times D \times [0,1] \to \mathbb{R}$. Then there exists an (ω, t)-affine support function $a_y : D \to \mathbb{R}$ of f at y such that $f - a_y$ is t-convex if and only if for all $u, v, x, z \in D$ and $s \in \{t, 1-t\}$ the function ω satisfies the conditions:*

(a) $\omega(y, y, t) = 0$,
(b) $\omega(x, z, t) = \omega(z, x, 1-t)$,
(c) $$\omega(u, z, s) + (1-s)\omega(v, z, s) - \omega(su + (1-s)v, z, s) \leq s\omega(u, v, s) - \omega(su + (1-s)z, sv + (1-s;)z, s).$$

For ω-convex functions a support theorem has the form

Theorem 28 ((V), Theorem 5) *Let D be a convex subset of a real linear space and let $y \in algint(D)$. Assume that $f : D \to \mathbb{R}$ is an ω-convex function, where $\omega : D \times D \times [0,1] \to \mathbb{R}$. Then there exists an ω-affine support function $a_y : D \to \mathbb{R}$ such that $f - a_y$ is convex if and only if for all $u, v, x, z \in D$ and all $s, t \in [0,1]$ the function ω satisfies the conditions:*

(i) $\omega(y, y, t) = 0$,
(ii) $\omega(x, z, t) = \omega(z, x, 1-t)$,
(iii) $$s\omega(u, z, t) + (1-s)\omega(v, z, t) - \omega(su + (1-s)v, z, t) \leq t\omega(u, v, s) - \omega(tu + (1-t)z, tv + (1-t;)z, s).$$

It turns out that the existence of an (ω, t)-affine (ω-affine) support at arbitrary point characterizes (ω, t)-convex (ω-convex) functions, so similarly as for convex functions, a given map has an (ω, t)-affine (ω-affine) support at arbitrary point if and only if it is (ω, t)-convex (ω-convex). Therefore, any (ω, t)-convex (ω-convex) function has the representation as the pointwise maximum of (ω, t)-affine (ω-affine) functions which are majorized by it.

As a consequence of support theorems we obtain the following representation theorem which is in the spirit of Ng's theorem.

Theorem 29 ((V), Theorem 7) *Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ (a real linear space) and let D be a t-convex (convex) set such that $algint_{\mathbb{Q}(t)}(D) \neq \emptyset$ ($algint(D) \neq \emptyset$). Assume that $f : D \to \mathbb{R}$ is an (ω, t)-convex (ω-convex) function, where $\omega : D \times D \times [0,1] \to \mathbb{R}$. Then there exist: t-convex (convex) function $h : D \to \mathbb{R}$ and (ω, t)-affine (ω-affine) function $a : D \to \mathbb{R}$ such that*

$$f(x) = a(x) + h(x) \quad \text{for } x \in D,$$

if and only if for some point $y \in algint_{\mathbb{Q}(t)}(D)$ ($y \in algint(D)$) ω satisfies the conditions (a)-(c) ((i)-(iii)).

The direct consequence of the above theorem is the following result

Theorem 30 ((V), Theorem 8) *Let X and D be as in the previous theorem, let $f : D \to \mathbb{R}$ be an (ω, t)-convex (ω-convex) function, and let $\omega : D \times D \times [0,1] \to \mathbb{R}$ satisfy the conditions* (a)-(c) ((i)-(iii)) *for some point $y \in algint_{\mathbb{Q}(t)}(D)$ ($y \in algint(D)$). If*

$$\omega(x,z,t) \geq 0 \text{ for } x,z \in D, \qquad (\omega(x,z,t) \geq 0 \text{ for } x,z \in D,\ t \in [0,1])$$

then f is a delta t-convex (delta-convex), i.e. there exist: t-convex (convex) functions $g, h : D \to \mathbb{R}$ such that

$$f(x) = g(x) - h(x), \quad x \in D.$$

If D is a convex subset of an inner product space then the function $\omega : D \times D \times [0,1] \to \mathbb{R}$ given by the formula

$$\omega(x,y,t) = ct(1-t)\|x-y\|^2$$

satisfies the conditions (a)-(c) and (i)-(iii). Therefore, using the main results of the paper (V) we get the support theorems for strongly t-convex functions (strongly convex) and approximate t-convex (approximate-convex) (for $\gamma = 2$). An analogous theorem for $t = \frac{1}{2}$ has been proved in the paper [10].

The next consequence of the main results of the paper (V) is a theorem which gives a characterization of inner product spaces. The following theorem gives the necessary and sufficient conditions under which a norm in a real linear space can be defined from an inner product.

Theorem 31 ((V), Theorem 14) *Let $(X, \|\cdot\|)$ be a real normed space. The following conditions are equivalent to each other:*

(α) *A map $\omega : D \times D \times [0,1] \to \mathbb{R}$ given by formula*

$$\omega(x,y,t) = ct(1-t)\|x-y\|^2, \quad x,y \in X$$

satisfies the inequalities (c) from Theorem 27 for some $c > 0$ and $t \in (0,1)$;

(β) *there exist a number $t \in (0,1)$ and a function $g : X \to \mathbb{R}$ such that*

$$\|x-y\|^2 = tg(x) + (1-t)g(y) - g(tx + (1-t)y) \quad \text{for } x,y \in X;$$

(γ) (X, $\|\cdot\|$) *is an inner product space.*

The most well-known result of this type is due to P. Jordan and J. von Neumann [56] which states that a real normed space is an inner product space if and only if it satisfies the parallelogram law i.e.

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for } x,y \in X.$$

A survey of analogous results can be found e.g. in monograph by D. Amir [7] (see also [6]).

The last application of Theorem 27 shows that not for all functions ω there exist the solutions to the inequality (4). Namely

Theorem 32 ((V), Theorem 15) *Let $(X, \|\cdot\|)$ be a real normed space, $t \in (0,1)$, $c > 0$. There is no function $f : X \to \mathbb{R}$ satisfying the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - c\|x-y\| \quad \text{for } x,y \in X.$$

An abstract version of the Hahn-Banach theorem and support theorem

A joint paper with Zsolt Páles (VI) was written mainly during my stay at the University of Debrecen. The aim of the paper was to prove a support theorem for a possibly wide class of maps, defined on abstract structures and taking values in partially ordered structures. A direct motivation to this paper was the paper (O7) (see part 5, p. 29).

Since the domains of the considered maps are general algebraic structures we introduce intuitive concepts of convex sets and extremal points for subsets of a given set without a linear structure. Let Γ and X be nonempty sets, and let $n : \Gamma \to \mathbb{N}$ be a given function. Assume that w is a given family of operations on X

$$w = \{w_\gamma : X^{n(\gamma)} \to X \mid \gamma \in \Gamma\}.$$

We say that $E \subseteq X$ is *w-convex* if

$$w_\gamma(E^{n(\gamma)}) \subseteq E \quad \text{for } \gamma \in \Gamma.$$

A subset $E \subseteq X$ is *w-extreme* if

$$w_\gamma^{-1}(E) \subseteq E^{n(\gamma)} \quad \text{for } \gamma \in \Gamma.$$

A point $p \in X$ is said to be *w-extreme* if the singleton $\{p\}$ is an w-extreme set. It is easy to check that the intersection of an arbitrary family of w-convex (w-extreme) sets is an w-convex (w-extreme) set and this allows us to introduce, for arbitrary subset A, an w-convex (w-extreme) hull of a set A as the smallest w-convex (w-extreme) set including the set A. We denote it by the symbol $conv_w(A)$ ($ext_w(A)$).

The notion of w-extreme hull leads as to introduce a counterpart of the notion of the relative interior and boundary in the following manner

Definition 3 A point $p \in X$ is said to be *w-internal* of X, if $ext_w(\{p\}) = X$. The set of w-internal points of X is called the *w-interior* of X and is denoted by $int_w(X)$. The complement of $int_w(X)$ is termed the *w-boundary* of X and is denoted by

$$\partial_w(X) := X \setminus int_w(X).$$

Generalizations of the Hahn-Banach theorem on the mapping taking values in partially ordered sets require additional assumptions regarding order structures. B. Rodriguez-Salinas and L. Bou [106] showed that sandwich type results can only be expected for ordered vector spaces where the intervals have the so-called binary intersection property.

In the literature the most frequently appearing assumption in the theorems of the Hahn-Banach type is the assumption that the range space is Dedekind complete i.e. every bounded from below subset has the greatest lower bound. R. J. Silverman and T. Yen [111] (see also [8], [18], [19], [53], [115], [116]) showed that if an order structure is induced by a linearly closed cone (closed in the so-called core topology) then the least upper bound property of the range space is indispensable, more precisely, an ordered vector space has the Hahn-Banach extension property if and only if it possesses the greatest lower bound property.

As for the range space we will assume that $(Y, \leq)$ is a partially ordered set in which every nonempty lower bounded chain has an infimum. If this is the case then we say that a partially ordered set $(Y, \leq)$ is *lower chain-complete*. Obviously, any Dedekind complete ordered space is a lower chain-complete but the converse is not true. Now, we give examples of partially ordered spaces which are lower chain-complete and not complete in the sense of Dedekind.

Let $(Y, +)$ be an abelian group. A nonempty subsemigroup S of the group Y is said to be *pointed* and *salient* if $0 \in S$ and $S \cap (-S) \subseteq \{0\}$, respectively. An arbitrary pointed and salient subsemigroup S induces a partial ordering $\leq_S$ on Y by letting

$$x \leq_S y \Leftrightarrow y - x \in S.$$

This partial order is compatible with the algebraic structure of Y in the sense that if $x \leq_S y$, then $x + z \leq_S y + z$ for each $z \in S$.

A triple $(Y, +, d)$ is called a *metric abelian group* if $(Y, +)$ is an abelian group, (Y, d) is a metric space and the metric is translation invariant, i.e.

$$d(x + z, y + z) = d(x, y) \quad \text{for } x, y, z \in Y.$$

In such a case, the metric induces a *pseudo norm* $\|\cdot\| : Y \to \mathbb{R}_+$ via the standard definition $\|x\|_d := d(x, 0)$. An important class of semigroups which are lower chain complete are the so-called additively controllable subsemigroups.

Definition 4 Let $(Y; +; d)$ be a metric abelian group. We say that a subsemigroup S of Y is *additively controllable* if there exists a continuous additive function $a : Y \to \mathbb{R}$ such that

$$\|y\|_d \leq a(y), \quad y \in S.$$

It turns out that each closed, additively controllable subsemigroup of a complete metric abelian group is a lower chain complete. In applications (e.g. in optimization theory) we consider the normed spaces and in a natural way a cone appears in the place of a subsemigroup. In the paper (VI) we have shown that any partially ordered space where the order is induced by a closed and convex cone such that $int(\mathcal{K}^\circ) \neq \emptyset$ is lower chain complete, where $\mathcal{K}^\circ$ denotes a dual cone with $\mathcal{K}$ i.e.

$$\mathcal{K}^\circ := \{\phi \in Y^\star : \phi(y) \geq 0, \ y \in \mathcal{K}\}.$$

An important example of a cone having this property is the so-called Lorentz cone:

$$\mathcal{K}_\varepsilon := \{(x, t) \in Y \times \mathbb{R} : \varepsilon\|x\| \leq t\}.$$

Now, we present the assumptions under which we have proved the main results of the paper (VI).

(H) Let X be a nonempty set, $(Y, \leq)$ partially ordered set, $\Gamma \neq \emptyset$, $n : \Gamma \to \mathbb{N}$, $w = \{w_\gamma : X^{n(\gamma)} \to X \mid \gamma \in \Gamma\}$ and $\Omega = \{\Omega_\gamma : Y^{n(\gamma)} \to Y \mid \gamma \in \Gamma\}$ are two given families of operations.

A family of operations $\{w_\gamma \mid \gamma \in \Gamma\}$ is said to be a *pairwise mutually distributive* if for all $\gamma, \beta \in \Gamma,\ k \in \{1, 2, \dots, n(\gamma)\}$ and all $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n(\gamma)}, y_1, \dots, y_{n(\beta)} \in X$

$$\begin{aligned} &w_\gamma(x_1\ , \dots, x_{k-1}, w_\beta(y_1, \dots, y_{n(\beta)}), x_{k+1}, \dots, x_{n(\gamma)}) \\ &= w_\beta(\, w_\gamma(x_1, \dots, x_{k-1}, y_1, x_{k+1}, \dots, x_{n(\gamma)}), \dots, \\ &\qquad w_\gamma(x_1, \dots, x_{k-1}, y_{n(\beta)}, x_{k+1}, \dots, x_{n(\gamma)})). \end{aligned}$$

The assumption of pairwise mutually distributivity is much more weaker than the pairwise commutativity which was needed for the setting of the Rodé's Theorem.

We say that a family of operations $\{w_\gamma \mid \gamma \in \Gamma\}$ is *reflexive* if, for all $\gamma \in \Gamma$,

$$w_\gamma(x, \dots, x) = x, \quad x \in X.$$

Under the hypothesis (H), given an w-convex set $D \subseteq X$, we say that $f : D \to Y$ is (w, Ω)-*convex on D* if it satisfies the functional inequality

$$f\big(w_\gamma(x_1, \dots, x_{n(\gamma)})\big) \leq \Omega_\gamma\big(f(x_1), \dots, f(x_{n(\gamma)})\big) \quad (\gamma \in \Gamma,\ x_1, \dots, x_{n(\gamma)} \in D).$$

If f satisfies the reversed inequality

$$\Omega_\gamma\big(f(x_1), \dots, f(x_{n(\gamma)})\big) \leq f\big(w_\gamma(x_1, \dots, x_{n(\gamma)})\big) \quad (\gamma \in \Gamma,\ x_1, \dots, x_{n(\gamma)} \in D),$$

then we say that it is (w, Ω)-*concave on D*. Finally, a function f is called (w, Ω)-*affine on D* if it satisfies the functional equation

$$f\big(w_\gamma(x_1, \dots, x_{n(\gamma)})\big) = \Omega_\gamma\big(f(x_1), \dots, f(x_{n(\gamma)})\big) \quad (\gamma \in \Gamma,\ x_1, \dots, x_{n(\gamma)} \in D).$$

The following theorem is a generalized version of the Hahn-Banach theorem for (w, Ω)-convex maps:

Theorem 33 ((VI), Theorem 4.3) *In addition to hypothesis (H), assume that*

(H1) $(Y, \leq)$ is a lower chain-complete partially ordered set.

(H2) The family w consists of pairwise mutually distributive operations.

(H3) The family Ω consists of pairwise mutually distributive operations such that, for all $\gamma \in \Gamma$, the operation Ω_γ is an order automorphism in each of its variables.

Let $f : X \to Y$ be an (w, Ω)-convex function and let $D \subseteq X$ be a nonempty w-convex subset of X such that $ext_w(D) = X$ and $f|_D$ is (w, Ω)-affine on D. Then there exists an (w, Ω)-affine function $g : X \to Y$ such that $g \leq f$ and $g|_D = f|_D$.

Assuming additionally, that the families of operations w and Ω are reflexive as a consequence we obtain a support theorem:

Theorem 34 ((VI), Corollary 4.4) *In addition to hypothesis (H), assume that*

(H1+) $(Y, \leq)$ *is a lower chain-complete partially ordered set.*

(H2+) The family w *consists of reflexive and pairwise mutually distributive operations.*

(H3+) The family Ω *consists of reflexive and pairwise mutually distributive operations such that, for all* $\gamma \in \Gamma$, *the operation* Ω_γ *is an order automorphism in each of its variables.*

Let $f : X \to Y$ *be an* (w, Ω)*-convex function. Then, for all* w*-interior point* $p \in X$, *there exists an* (w, Ω)*-affine function* $g : X \to Y$ *such that* $g \leq f$ *and* $g(p) = f(p)$.

The next consequence is a support theorem for subadditive mappings defined on abstract structures. This theorem generalizes the theorem of Berz [15] and Theorem 26 on support for delta-subadditive maps.

Theorem 35 ((VI), Corollary 4.5) *Let* $(X, +)$ *be an abelian semigroup, and let* $(Y, +, d)$ *be a complete metric abelian group equipped with an ordering* $\leq_S$ *generated by a closed pointed additively controllable semigroup* $S \subseteq Y$. *Let* $f : X \to Y$ *be a subadditive map, i.e.*

$$f(x + y) \leq_S f(x) + f(y) \quad \text{for } x, y \in X. \tag{1}$$

Assume that $p \in X$ *possesses the following two properties:*

(i) for all $n \in \mathbb{N}$, $f(np) = nf(p)$;

(ii) for all $x \in X$, *there exist* $y \in X$ *and* $n \in \mathbb{N}$ *such that* $x + y = np$.

Then there exists an additive function $g : X \to Y$ *such that* $g \leq_S f$ *and* $g(p) = f(p)$.

We recall here the definition of $\frac{1}{2}$-convex set. Let $(G, +)$ be a uniquely 2-divisible abelian group. A subset X of group G is said to be $\frac{1}{2}$-convex if for arbitrary $x, y \in X$,

$$\frac{1}{2}(x + y) \in X.$$

It easily follows by induction, that if X is $\frac{1}{2}$-convex then for all $n \in \mathbb{N}$ and for all $k \in \{0, 1, 2, \ldots, 2^n\}$ we have

$$\frac{k}{2^n}x + (1 - \frac{k}{2^n})y \in X.$$

Definition 5 *Let* $(G, +)$ *be a uniquely* 2*-divisible abelian group,* $X \subseteq G$. *We say that* $p \in G$ *is a* relative algebraic interior point *of the set* X, *(we write* $p \in ri(X)$*) if for all* $x \in X$ *there exists* $n \in \mathbb{N}$ *such that* $p + \frac{1}{2^n}(p - x) \in X$.

Using these concepts we can formulate another support theorem

Theorem 36 ((VI), Theorem 4.8) *Let X be a $\frac{1}{2}$-convex subset of a uniquely 2-divisible abelian group $(G,+)$, and let $(Y,+,d)$ be a complete metric abelian group equipped with an ordering $\leq_S$ generated by a closed pointed additively controllable semigroup $S \subseteq Y$. Moreover, assume that $n \geq 2$, $a_1, \dots, a_n : G \to G$ and $A_1, \dots, A_n : Y \to Y$ are two families of additive maps with the following additional properties:*

(i) $a_i \circ a_j = a_j \circ a_i$ *and* $A_i \circ A_j = A_j \circ A_i$, *for all* $i, j = 1, \dots, n$;

(ii) $a_1 + \ldots + a_n = id_G$ *and* $A_1 + \ldots + A_n = id_Y$;

(iii) $a_1(X) + \ldots + a_n(X) \subseteq X$;

(iv) A_i *is bijective with* $A_i(S) = S$ *for all* $i \in \{1, \dots, n\}$.

Let $f : X \to Y$ satisfy, for all $x_1, \dots, x_n \in X$, the following convexity type inequality

$$f\big(a_1(x_1) + \cdots + a_n(x_n)\big) \leq_S A_1\big(f(x_1)\big) + \cdots + A_n\big(f(x_n)\big).$$

Then, for every $p \in ri(X)$, there exists a function $g : G \to Y$ such that $g \leq_S f$, $g(p) = f(p)$ and, for all $x_1, \dots, x_n \in X$, the following functional equation holds:

$$g\big(a_1(x_1) + \cdots + a_n(x_n)\big) = A_1\big(g(x_1)\big) + \cdots + A_n\big(g(x_n)\big).$$

As a direct consequence of the above theorem (by specifying additive mappings and considering an order generated by the Lorentz cone) we get the main result from the paper (O7) which was the motivation to the above considerations.

5. Presentation of other research achievements

(a) The list of publications not included in the habilitation thesis:

(O1) A. Olbryś, *On the measurability and the Baire property of t-Wright convex functions*, Aequationes Math. 68 (2004), no. 1-2, 28-37.

(O2) A. Olbryś, *Some conditions implying the continuity of t-Wright convex functions*, Publ. Math. Debrecen 68 (2006), no. 3-4, 401-418.

(O3) A. Olbryś, *A characterization of $(t_1, \dots, t_n)$-Wright affine functions*, Comment. Math. (Prace Mat.) 47 (2007), no. 1, 47-56.

(O4) A. Olbryś, *On the boundedness, Christensen measurability and continuity of t-Wright convex functions*, Acta Math. Hungar. 141 (2013), no. 1-2, 68-77.

(O5) M. Lewicki, A. Olbryś, *On non-symmetric t-convex functions*, Math. Inequal. Appl. 17 (2014), no. 1, 95-100.

(O6) A. Olbryś, *On some inequalities equivalent to the Wright-convexity*, J. Math. Inequal. 9 (2015), no. 2, 449–461.

(O7) A. Olbryś, *A support theorem for delta (s,t)-convex mappings*, Aequationes Math. 89 (2015), no. 3, 937–948.

(O8) A. Olbryś, *On delta Schur-convex mappings*, Publ. Math. Debrecen 86 (2015), no. 3-4, 313-323.

(O9) A. Olbryś, *On separation by h-convex functions*, Tatra Mt. Math. Publ. 62 (2015), 105-111.

(O10) A. Olbryś, *Representation theorems for h-convexity*, J. Math. Anal. Appl, 426 (2015), no. 2, 986–994.

(O11) A. Olbryś, T. Szostok, *Inequalities of the Hermite–Hadamard Type Involving Numerical Differentiation Formulas*, Results. Math. 67 (2015), no. 3-4, 403–416.

(O12) W. Fechner, A. Olbryś, *Systems of Inequalities Characterizing Ring Homomorphisms*, J. Funct. Spaces 2016, Art. ID 8069104, 5 pp.

(O13) A. Olbryś, *On the* $\mathbb{K}$*-Riemann integral and Hermite-Hadamard inequalities for* $\mathbb{K}$*-convex functions*, Aequationes Math. 91 (2017), no. 3, 429-444.

(O14) A. Olbryś, *On a problem of T. Szostok concerning the Hermite-Hadamard inequalities*, arXiv:1808.06524 [math.CA].

(b) A description of the scientific output contained in the papers listed in item 5(a)

The paper (O3) is a part of PhD thesis. I give in it a characterization of t-Wright affine functions of higher orders. This paper generalizes the results of Lajkó [72], who characterized t-Wright affine functions defined on interval and taking real values. The results of Lajkó was generalized in two directions. First we gave a form of solutions to a much more general functional equation than Lajkó, secondly the considered maps are defined and takes values in abstract structures.

The papers (O1), (O2), (O4) concern the continuity problem for t-Wright convex functions. In the theory of functional equations and inequalities one of the most important areas of research is the problem of "improving regularity" of solutions. The goal is to show that a function satisfying a given functional equation or inequalities under possibly weak regularity assumptions has a higher regularity. The most famous result of this type is a theorem proved by Bernstein and Doetsch [14] in 1915 which states that each convex function in the sense of Jensen, bounded above on a nonempty open interval is convex, so in particular, continuous at any internal point of the domain. On the other hand, Sierpiński's theorem states that each Lebesgue measurable and Jensen convex function is convex. Many generalizations of mentioned results are known, see for example [34], [42], [62], [71], [80]. The papers (O1) and (O2) were part of the PhD thesis.

In the paper (O1) the t-Wright convex functions defined on interval are considered. It has been proved that both measurability in the sense of Lebesgue and Baire implies the continuity of such functions. In the paper (O2) it is shown that every t-Wright convex function, continuous at least at one point of an open and convex subset of a real linear-topological space is continuous at every point. This result is a generalization of Kominek's Theorem proved in [61] for functions defined on interval. The main result of the paper (O2) states that any t-Wright convex function which restriction to the "large set" in the sense of measure or category, is lower semicontinuous, is continuous and convex.

The paper (O4) consists of two parts. In the first part we discuss a connection between the local boundedness of t-Wright convex functions and continuity. It is known that one side local boundedness from above or below (even global) does not imply the continuity of these functions. The question about the suitable version of Bernstein and Doetsch Theorem for bounded function (from above and below) is natural. The following theorem is the main result of the first part of this paper.

Theorem 37 ((O4), Theorem 5) *Let D be an open and convex subset of a real linear-topological space, and let $f : D \to \mathbb{R}$ be a t-Wright convex function. If f is locally bounded at some point then it is continuous.*

The above theorem motivates to study the following class of sets:

$$C_t(D) := \{\, T \subseteq D \mid \text{every } t\text{-Wright convex function } f : D \to \mathbb{R} \text{ bounded on } T, \text{ is continuous}\}$$

In the paper it is shown that some sets that are "smaller" than the set with nonempty interior (in the measure and category sense) belong to the class $C_t(D)$. An analogous class of sets for convex functions in the sense of Jensen was introduced by Ger and Kuczma in the paper [42]. Many facts about this concept can be found in the monograph [71].

In the second part of the paper we proved that also Christensen measurability implies convexity and continuity of t-Wright convex functions. An analogous result for convex functions in the sense of Jensen ($t = \frac{1}{2}$) was done by P. Fischer and Z. Słodkowski in [34].

Joint paper with Michał Lewicki (O5) concerns non-symmetric t-convex functions. The definition of these functions was proposed by Páles who modified the definition of t-convex functions in the following manner:

Definition 6 Let $t \in (0, 1)$ be a given number. A function $f : I \to \mathbb{R}$ is said to be a *non-symmetric t-convex* (where $I \subseteq \mathbb{R}$ is an interval) if

$$f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y) \quad \text{for such } x, y \in I, \text{ that } x < y.$$

Since every t-convex function is convex in the sense of Jensen, the natural question is whether this property have also non-symmetric t-convex functions. This problem was posed by Páles orally, during the conferences of ISFE series. In the discussed paper we showed that every non-symmetric t-affine function i.e. a solution to the corresponding functional equation is a t-affine function, i.e. it satisfies this equation for all $x, y \in I$. In particular, it satisfies this functional equation for $t = \frac{1}{2}$.

On the other hand we gave a negative solution to the Páles problem by constructing two examples. The first is an example of a non-symmetric t-convex function which is non-symmetric t-concave, the second is an example of a non-symmetric t-convex function which is concave in the sense of Jensen. These examples show also, that non-symmetric t-convex functions do not have non-symmetric t-affine support functions at an arbitrary point.

In the paper (O6) convex functions in the sense of Wright are studied. Several characterizations of these functions are given. It turns out that these functions can be described in term of the second symmetric difference. I have shown that the convexity in the sense of Wright of function $f : D \to \mathbb{R}$ is equivalent to the convexity of function

$$[0,1] \ni t \longrightarrow f(tx + (1-t)y) + f((1-t)x + ty),$$

for all $x, y \in D$ and the function

$$D_y \ni x \longrightarrow f(x) + f(2y - x),$$

for all $y \in D$, where $D_y = D \cap (2y - D)$, and D is a convex set. Using the above characterization I obtained, without any additional assumptions, suitable versions of the Hermite-Hadamard inequalities. In the literature there are known some results concerning the Hermite-Hadamard inequalities for Wright-convex functions, unfortunately, the authors of all these articles assume that the considered Wright-convex functions are measurable, so in particular, continuous and convex, and this leads to the classical Hermite-Hadamard inequalities.

The paper (O7) concerns to the so-called delta (s,t)-convex maps. I give in it a common generalization of delta-convex maps introduced by Veselý and Zajíček in [120] and (s,t)-convex functions introduced by Kuhn in the paper [70]. Assume that $s, t \in (0,1)$ are given numbers and let X, Y be a real Banach spaces.

Definition 7 Let $D \subseteq X$ be a convex set. A map $F : D \to Y$ is said to be a *delta (s,t)-convex* with a control function $f : D \to \mathbb{R}$, if for all $x, y \in D$ the inequality

$$\begin{aligned}\|tF(x) + (1-t)F(y) - &F(sx + (1-s)y)\| \\ &\le tf(x) + (1-t)f(y) - f(sx + (1-s)y),\end{aligned}$$

holds. In the case, where $s = t$, we say that F is a *delta t-convex* with a control function f; if $t = \frac{1}{2}$, then F is called *delta convex in the sense of Jensen* with a control function f.

I have shown, using Daróczy and Páles identity [29] that every delta (s,t)-convex map has to be delta convex in the sense of Jensen. In particular, any delta t-convex map is a delta convex in the sense of Jensen, and from this fact we can infer that in the class of continuous functions the concepts of delta-convexity and delta t-convexity are equivalent. In the case, where $t = \frac{1}{2}$ this theorem was proved by Veselý and Zajiček in [120]. The main result of (O7) is the following support theorem for delta (s,t)-convex maps:

Theorem 38 ((O7), Theorem 4) *Let $D \subseteq X$ be an open and convex set, and let $F : D \to Y$ be a delta (s,t)-convex map with a control function $f : D \to \mathbb{R}$. Then for an arbitrary point $y \in D$ there exist: (s,t)-affine maps $A_y : D \to Y$ and $a_y : D \to \mathbb{R}$ such that*

$$\|F(x) - A_y(x)\| \leq f(x) - a_y(x) \quad \text{for } x \in D$$

and

$$A_y(y) = F(y), \quad a_y(y) = f(y).$$

It turns out that the existence of an (s,t)-affine support function at an arbitrary point characterizes (s,t)-convex maps.

The paper (O8) was devoted to delta Schur convex maps. I generalize the definition of preorder on the Cartesian product of arbitrary real linear spaces and introduce a common generalization of delta-convex maps and Schur convex functions.

Definition 8 Let X be a real linear space. We say that a vector $x = (x_1, ..., x_n) \in X^n$ is *majorized* by vector $y = (y_1, ..., y_n) \in X^n$, written $x \prec y$, if there exists a doubly stochastic matrix $S \in \mathbb{R}^n_n$ such that

$$(x_1, ..., x_n) = (y_1, ..., y_n)S.$$

A common generalization of the concepts of convex functions in the sense of Schur and delta-convex mappings gives the following definition:

Definition 9 Let X and Y be real normed spaces, and let $D \subseteq X$ be a convex set. We say that a map $F : D^n \to Y$ is *delta convex in the sense of Schur* with a control function $f : D^n \to \mathbb{R}$ if for all $x, y \in D^n$ we have

$$x \prec y \Longrightarrow \|F(x) - F(y)\| \leq f(y) - f(x).$$

It is easy to check that in the case, where $Y = \mathbb{R}$ a given map is a delta convex in the sense of Schur if and only if it is a difference of two convex function in the sense of Schur. In this case a delta convex functions in the sense of Schur form the smallest linear subspace containing the cone of all convex functions in the sense of Schur. Therefore this concept is a natural generalization to the case of mappings taking the vector values of functions which are the difference of two convex functions in the sense of Schur. The purpose of the paper was to generalize the result of Ng [81] i.e. providing a characterization of delta convex maps in the sense of Schur of the form $H(x_1, ..., x_n) = \sum_{j=1}^{n} F(x_j)$. Using a support theorem [(O7), Theorem 1] I prove the following counterpart of the Ng's theorem (see Theorem 1, p. 4)

Theorem 39 ((O8), Theorem 5) *Let X and Y be two real Banach spaces, and let $D \subseteq X$ be an open and convex set. Let $F : D \to Y$, $f : D \to \mathbb{R}$ and put $G(x_1, \ldots, x_n) = \sum_{j=1}^{n} F(x_j)$, $g(x_1, \ldots, x_n) = \sum_{j=1}^{n} f(x_j)$. Then the following conditions are pairwise equivalent:*

(i) *G is delta convex in the sense of Schur with a control function g, for some $n \geq 2$;*

(ii) *G is delta convex in the sense of Schur with a control function g, for every $n \geq 2$;*

(iii) *F is delta convex in the sense of Wright with a control function f i.e.*

$$\|F(x) + F(y) - F(tx + (1-t)y) - F((1-t)x + ty)\| \\ \leq f(x) + f(y) - f(tx + (1-t)y) - f((1-t)x + ty)$$

for all $x, y \in D$, $t \in [0, 1]$;

(iv) *F has the form*

$$F(x) = W(x) + A(x), \quad x \in D,$$

where $W : D \to Y$ is a delta-convex map, and $A : X \to Y$ is an additive map.

The last theorem of the paper generalizes the Ng's theorem on maps taking values in vector spaces for more general sum of the form $\sum_{j=1}^{n} F_j(x_j)$:

Theorem 40 ((O8), Theorem 6) *Let X and Y be two real Banach spaces, and let $D \subseteq X$ be an open and convex set. If $F_j : D \to Y$ and $f_j : D \to \mathbb{R}$ for $j = 1, ..., n$, then the map $G(x_1, \ldots, x_n) = \sum_{j=1}^{n} F_j(x_j)$ is delta convex in the sense of Schur with a control function $g(x_1, \ldots, x_n) = \sum_{j=1}^{n} f_j(x_j)$ if and only if there exist constants $C_1, ..., C_n \in Y$, additive map $A : X \to Y$ and delta-convex map $W : D \to Y$ such that*

$$F_j(x) = A(x) + W(x) + C_j \quad \textit{for } j = 1, ..., n \textit{ and } x \in D.$$

The papers (O9), (O10) relate to h-convex functions. The definition of these functions was introduced by S. Varošanec in [118] in the following way:

Definition 10 Let D be a convex subset of a real linear space, $h : [0, 1] \to \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be *h-convex*, if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \quad \text{for } x, y \in D,\ t \in [0, 1].$$

The concept of h-convex functions generalizes a concept of convex functions ($h(t) = t$), s-convex in the sense of Breckner ($h(t) = s^t$) [21], so-called Godunova-Levin functions ($h(t) = \frac{1}{t}$, for $t \in (0, 1)$) [43] and P-functions ($h(t) = 1$) [94].

In the paper (O9) I give the necessary and sufficient conditions under which two functions can be separated by h-convex function, for multiplicative function h. This theorem generalizes on infinite dimensional spaces the Baron-Matkowski-Nikodem theorem which gives the necessary and sufficient conditions for the separation by a convex functional.

In the paper (O10) I study h convex functions satisfying the condition

$$h(t) + h(1-t) = 1 \quad \text{for } t \in [0,1]. \tag{5}$$

If the above condition holds then an h-convex function may take arbitrary values, otherwise it is either non-negative or non-positive. The main result of the paper (O10) is the following characterization of h-convex functions satisfying the condition (5):

Theorem 41 (O10), Theorem 5) *Let D be an algebraically open and convex subset of a real linear space and let $h : [0,1] \to \mathbb{R}$ satisfy the condition* (5)*. Then $f : D \to \mathbb{R}$ is an h-convex function if and only if f is a constant function or $h(t) = t,\ t \in [0,1]$. In particular, f is convex function.*

Joint paper with Tomasz Szostok (O11) was devoted to the new method of proving of inequalities using the Levin-Stečkin theorem [73], which provides the necessary and sufficient conditions under which for any continuous and convex function $f : [a,b] \to \mathbb{R}$ the inequality

$$\int_a^b f(x)dF_1(x) \le \int_a^b f(x)dF_2(x),$$

holds, where $F_1, F_2 : [a,b] \to \mathbb{R}$ are functions of bounded variation. Many classic inequalities fulfilled by the convex functions can be easily obtained by specifying the functions F_1 and F_2 in the Levin-Stečkin theorem. In particular, the Hermite-Hadamard inequalities which can be rewritten in the form

$$f\Big(\frac{x+y}{2}\Big) \le \frac{F(y)-F(x)}{y-x} \le \frac{f(x)+f(y)}{2},$$

where $F' = f$. In the paper we obtain more general inequalities, replacing the middle term $\frac{F(y)-F(x)}{y-x}$ by expressions of numerical differentiation of the form

$$\frac{\sum_{i=1}^n a_i F(\alpha_i x + (1-\alpha_i)y)}{y-x},$$

where $\sum_{i=1}^n a_i = 0$.

Joint paper with Włodzimierz Fechner (O12) concerns the characterizations of ring homomorphism through the system of functional inequalities. The first result of this type was done by M. Rădulescu in [98]. Let X be a compact Hausdorff topological space, and let $C(X)$ be the space of all continuous real valued functions defined on X and equipped with the supremum norm. Rădulescu in the paper [98] showed that if an operator $T : C(X) \to C(X)$ satisfies the system of inequalities

$$\begin{cases} T(f+g) \ge T(f) + T(g) \\ T(f \cdot g) \ge T(f) \cdot T(g) \end{cases} \tag{6}$$

for all $f, g \in C(X)$, then there exists a clopen subset $B \subseteq X$ and a continuous function $\phi : X \to X$ such that

$$T(f) = \chi_B \cdot f \circ \phi,$$

where χ_B denotes the characteristic function of the set B. In particular, T is linear, multiplicative, and continuous. Ercan in the paper [31] has shown that the assumption that X is a compact Hausdorff space may be dropped. The system (6) has been extensively studied by the following authors: J. X. Chen, Z. L. Chen [25], J. Dhombres [30], W. Fechner [32, 33], I. Gusić [44] anf Volkmann [123, 124].

In this paper we considered more general system of inequalities for operators defined on rings and taking values in ordered rings. We showed, that under some technical assumptions on rings, operators $U, T : \mathcal{P} \to \mathcal{R}$, satisfy the system

$$\begin{cases} T(f+g) \geq T(f) + T(g) \\ U(f \cdot g) \geq U(f) \cdot U(g) \end{cases}$$

and the inequality $U \leq T$ for all $f, g \in \mathcal{P}$ if and only if $U = T$ is a ring homomorphism.

The next result of this type states that if additionally $1 \in \mathcal{P}$, then operators $T, U : \mathcal{P} \to \mathcal{R}$ satisfy the system:

$$\begin{cases} T(f+g) \geq U(f) + T(g) \\ U(f \cdot g) \geq T(f) \cdot T(g) \end{cases}$$

for all $f, g \in \mathcal{P}$ if and only if $U = T$ is a ring homomorphism.

As an application of theorems of these type we give a sufficient condition for the separation of two operators $T, U : \mathcal{B}(X) \to \mathcal{B}(X)$ (where $\mathcal{B}(X)$ denotes the space of all bounded real valued functions defined on X) by an operator which is additive and multiplicative simultaneously.

The purpose of the paper (O13) was to prove a counterpart of the Hermite-Hadamard inequalities for convex functions in the sense of Jensen. However, these functions can be irregular (e.g. a discontinuous at every point, nonmeasurable in particular, non-integrable) it was necessary to generalize the concept of Riemann integral on wider class of maps.

For a subfield $\mathbb{K} \subseteq \mathbb{R}$ let us define a generalized interval in the following way

$$[a, b]_{\mathbb{K}} := \{\alpha a + (1-\alpha)b : \ \alpha \in [0, 1] \cap \mathbb{K}\}.$$

Next, for a given function $f : [a, b]_{\mathbb{K}} \to \mathbb{R}$ we define lower and upper Darboux sums, lower and upper integral as for classic Riemann integral restricted to the set $[a, b]_{\mathbb{K}}$. If the lower and upper integrals are equal we say that f is $\mathbb{K}$-*Riemann integrable* and this common value we denote by the symbol

$$\int_a^b f(x) d_{\mathbb{K}} x.$$

Clearly, if $\mathbb{K}_1 \subseteq \mathbb{K}_2$, then every $\mathbb{K}_2$-Riemann integrable function is also $\mathbb{K}_1$-Riemann integrable but the converse is not true. In particular, every Riemann integrable function in the classical sense is $\mathbb{K}$-Riemann integrable for any $\mathbb{K} \subseteq \mathbb{R}$ and these integrals are equal. This concept indeed generalizes Riemann integral. In the paper I have proved number of properties of $\mathbb{K}$-Riemann integral, I discuss the connection with the $\mathbb{K}$-derivative, introduced by Z. Boros and Páles in [20] and give the classes of functions integrable in generalized sense. It turns out that any $\mathbb{K}$-convex function $f : I \to \mathbb{R}$, i.e.

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \quad \text{for } x, y \in I, \text{ and } \alpha \in [0,1] \cap \mathbb{K},$$

is a $\mathbb{K}$-Riemann integrable on an arbitrary interval $[a,b]_{\mathbb{K}} \subseteq I$. Especially, any $\mathbb{K}$-linear function $g : \mathbb{R} \to \mathbb{R}$ i.e. additive and $\mathbb{K}$-homogeneous has this property, moreover,

$$\int_a^b g(x) d_{\mathbb{K}} x = g\Big(\frac{a+b}{2}\Big)(b-a).$$

The main result of the paper states that if $f : I \to \mathbb{R}$ is a convex function in the sense of Jensen then for all $a, b \in I,\ a < b,$, f is a $\mathbb{Q}$-Riemann integrable on the set $[a,b]_{\mathbb{Q}}$, moreover,

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x) d_{\mathbb{Q}} x \leq \frac{f(a)+f(b)}{2}.$$

The concept of a generalized Riemann integral turned out to be a very useful tool. I also used them in the paper (O14). In this paper I gave a solution to the problem posed by Tomasz Szostok [112], who asked about the solutions $f, F : (a,b) \to \mathbb{R}$ to the system of inequalities

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{F(y)-F(x)}{y-x} \leq \frac{f(x)+f(y)}{2} \quad \text{for } x, y \in (a,b),\ x \neq y. \tag{7}$$

The following theorem provides a solution to the problem of Szostok:

Theorem 42 ((O14), Theorem 1) *The functions $F, f : (a,b) \to \mathbb{R}$ satisfy the system of inequalities* (7) *if and only if f is a convex function, and F is a primitive function of f, i.e. $F' = f$.*

The above result can be understood as a regularity phenomenon. The solutions to the system of functional inequalities turn out to be regular without any regularity assumptions. In the literature there are known several results of these type for functional equations but this phenomenon is very rare for the solutions to the functional inequalities. As we known, the inequality defining the convexity, for functions defined on an open and convex subset of a finite-dimensional real linear space has only continuous solutions. The same effect were obtained by the authors in mentioned papers [31], [98] for a system of functional inequalities characterizing the ring homomorphisms. A characterization of functions satisfying the so-called Shannon inequality were obtained by J. Aczél and A. M. Ostrowski in the paper [4] (see also [3], p. 116). They have shown that the function $h : [0,1] \to \mathbb{R}$ satisfies the functional inequality

$$\sum_{k=1}^{n} p_k h(p_k) \le \sum_{k=1}^{n} p_k h(q_k),$$

for all such $p_1, \dots, p_n;\ q_1, \dots, q_n \in (0,1)$ that $\sum_{k=1}^{n} p_k = \sum_{k=1}^{n} q_k = 1$ if and only if there exist constants $b, c \in \mathbb{R},\ c \le 0$ such that

$$h(p) = c \log p + b \quad \text{for } p \in (0,1).$$

For functional equations the regularity phenomenon is characteristic for equations derived from the mean values theorems. In 1985 Aczél proved that the functions $f, F : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$F(y) - F(x) = (y - x) f\left(\frac{x+y}{2}\right)$$

if and only if $F(x) = cx^2$ and $f(x) = 2cx$. Results in this spirit can be found in the papers [47], [54], [59], [60], [100], [108] and in monograph [99].

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