Summary of scientific achievements

1. Name and surname: Barbara Przebieracz
2. Academic degrees:
   (a) Master of Science in Mathematics;
   degree awarded by the University of Silesia, Katowice, on 12th of June 2003
   title of Master's Thesis: *The Hausdorff dimension of cartesian product of sets*,
   supervisor: *prof. dr hab. Andrzej Lasota.*
   (b) Doctor's degree in Mathematical Sciences;
   degree awarded by the University of Silesia, Katowice, on 3rd of July 2007
   title of Ph.D. Thesis: *Functions close to the iterable ones*,
   supervisor: *dr hab. Witold Jarczyk.*
3. Employment in academic institutions:
   Assistant Professor; Real Analysis Section, Institute of Mathematics, University of Silesia; from 1.09 2007.
4. Indication of scientific achievement
   Title of the habilitation thesis:

"On an Ulam's problem"

Papers constituting the habilitation thesis:

[A] Barbara Przebieracz, *On the stability of the translation equation and dynamical systems*, Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods 75, No. 4, (2012), 1980–1988.

[B] Barbara Przebieracz, *The Hyers theorem via the Markov-Kakutani fixed point theorem*, J. Fixed Point Theory Appl. 12, (2012), 35–39 .

[C] Barbara Przebieracz *A characterization of the approximate solutions of the translation equation*, J. Difference Equ. Appl. 21 (2015), no. 11, 1058–1067.

[D] Barbara Przebieracz, *An application of the common fixed point theorems to the theory of stability of functional equations,* Fixed Point Theory 16 (2015), no. 1, 185–190.

[E] Barbara Przebieracz, *Remarks on Farah's Theorems*, Results in Math., 72(4) (2017), 1959–1966.

[F] Roman Badora & Barbara Przebieracz, *On approximate group homomorphisms*, J. Math. Anal. Appl. 462(1) (2018), 505–520.

[G] Roman Badora, Tomasz Kochanek & Barbara Przebieracz, *Approximate homomorphisms on lattices*, Archiv der Mathematik, DOI: 10.1007/s00013-018-1182-0

Contents

1. Description of the field and motivation

1.1 Introduction

In 1940 S. M. Ulam gave a lecture to The Mathematics Club of the University of Wisconsin, where he presented a number of unsolved problems. One of them is considered to be the starting point of the theory of stability of functional equations[1]. It can be formulated as follows. Let G be a group and (H, d) a metric group. Does there exist for every $\varepsilon > 0$ a $\delta > 0$ such that for every $f : G \to H$ satisfying

$$d(f(xy), f(x)f(y)) \leq \delta, \quad \text{for all} \;\; x, y \in G,$$

there exists a homomorphism $a : G \to H$ such that

$$d(f(x), a(x)) \leq \varepsilon, \quad \text{for all} \;\; x \in G?$$

Twenty years after giving this lecture S. M. Ulam published a book "Problems in Modern Mathematics" [117], in which he formulated the problem of stability. First section of the sixth chapter: "Some Questions in Analysis" is entitled "Stability". There he writes: ”For very general functional equations one can ask the following question. When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation? Similarly, if we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality lie near to the solutions of the strict equation?”

Even more generally the problem of stability was formulated (while investigating stability of isometries) in 1978 by P. M. Gruber [41], who reformulated the Ulam's problem in this way: “Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this object by objects, satisfying the property exactly?”

At this point it is not possible to present a complete picture of stability theory of functional equations. However, I would like to specify the main lines of research that emerged from the question of S. Ulam, highlight the relationship of this theory with other branches of mathematics and mention what are the most popular tools used in the study of such issues.

1.2 Hyers Theorem and the discussion concerning the domain

Less than a year after the lecture of S. Ulam, D. H. Hyers obtained the first important result related to this problem. In his paper [42] one can find the following theorem:

[1]Even earlier, in 1924, the similar problem was considered by G. Pólya i G. Szegö (see [94]) in some special case.

Theorem 1.1 (D. H. Hyers)**.** *Let X and Y be Banach spaces and $f : X \to Y$ satisfy the condition*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \quad x, y \in X, \tag{1.1}$$

with some $\varepsilon > 0$. Then there exists a (unique) additive function $a : X \to Y$ satisfying the inequality

$$\|f(x) - a(x)\| \leq \varepsilon, \quad x \in X.$$

Therefore, in the case where G and H are Banach spaces, the answer to Ulam's question is positive (with $\delta = \varepsilon$); we say then that the Cauchy equation $f(x+y) = f(x) + f(y)$ is stable. In addition, looking at Hyers' proof, we can immediately see that X can be replaced by any additive semigroup. In the discussion under what assumption one can get a similar approximation as in the Hyers Theorem let's note that in [26] G. L. Forti showed that the Cauchy equation is not stable on the free group generated by two elements. L. Székelyhidi [106] (see also [105]) showed that stability can have place also in the case of non-commutative domain – it is enough to assume that the domain is an amenable semigroup. However, this assumption turned out to be too strong, as proved J. Lawrence [27]. Further research in this direction conducted, among others, L. Giudici, whose unpublished results can be found in a survey by G. L. Forti [28].

1.3 Discussion concerning target space

Similarly the assumptions concerning the target space in Hyers Theorem were discussed. G. L. Forti and J. Schwaiger [29] proved the following theorem:

Theorem 1.2 (G. L. Forti, J. Schwaiger)**.** *Let G be a commutative group with an element of infinite order and let Y be a normed space. Then the following implication holds true: if for every function $f : G \to Y$ satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \;\; x, y \in G,$$

there exists a homomorphism $a : G \to Y$ such that

$$\|f(x) - a(x)\| \leq \varepsilon, \;\; x \in G,$$

then Y is complete.

Another line of research was indicated by L. Székelyhidi in [107], its culmination is the following theorem proved by Z. Gajda [31].

Theorem 1.3 (Z. Gajda)**.** *Suppose that the Hyers Theorem holds true for complex functions defined on a semigroup S and let Y be a sequentially complete linear-topological*

Hausdorff space. Then, if $f : S \to Y$ and

$$S \times S \ni (x, y) \longrightarrow f(x + y) - f(x) - f(y)$$

is bounded then there exists an additive function $a : S \to Y$ such that the difference $f - a$ is bounded too.

1.4 Methods of proving stability

In the theory of the stability of functional equations several techniques have been developed. They allow us to study the stability of some types of classical functional equations. The technique based on the so-called Hyers sequences is the most popular and often used. It refers to the original proof of Hyers [42].

A second technique was introduced by J. A. Baker in [6]. Baker based the proof of the following theorem concerning stability of a nonlinear functional equation on the Banach Contraction Principle.

Theorem 1.4 (J. A. Baker)**.** *Let T be a nonempty set, (Y, ρ) a complete metric space, $\phi : T \to T$, $F : T \times Y \to Y$, $0 \le \lambda < 1$ and let*

$$\rho(F(T, u), F(t, v)) \le \lambda\rho(u, v), \;\; t \in T, \; u, v \in Y.$$

Then, for every function $f : T \to Y$ satisfying the condition

$$\rho(f(t), F(t, f(\phi(t)))) < \varepsilon, \;\; t \in T,$$

with some $\varepsilon \ge 0$, there exists a unique function $f_0 : T \to Y$ such that

$$f_0(t) = F(t, f_0(\phi(t))), \;\; t \in T$$

and

$$\rho(f(t), f_0(t)) < \varepsilon/(1 - \lambda), \;\; t \in T.$$

Later the fixed point method was successfully used in the theory of stability (see for example [18]).

Another technique, based on invariant means, was proposed by L. Székelyhidi in [106], [105]. Namely, L. Székelyhidi proved Hyers Theorem for complex mappings defined on an amenable semigroup.

However, it seems that the diversity of the issues under consideration does not allow for the development of any universal techniques in the study of Ulam-type problems.

1.5 Different "types of stability"

Considering Ulam's problem for a functional equation of exponential function:

$$f(x + y) = f(x)f(y),$$

resulted in the following statement:

Theorem 1.5 (J. A. Baker [5]). *If S is a semigroup and $f : S \to \mathbb{C}$ satisfies*

$$|f(x+y) - f(x)f(y)| \leq \varepsilon, \quad x, y \in S$$

with some $\varepsilon \geq 0$, then either f is bounded (by $(1+\sqrt{1+4\varepsilon})/2$) or it is an exponential function.

In this case we say that the functional equation is superstable. R. Ger in [34], [35] explained that such unusual behavior is a consequence of mixing addition and multiplying in $\mathbb{C}$. [2]

Superstability is only one of many possible "types of stability" in the sense of Ulam. Others are, for example, b-stability, uniform b-stability, inverse stability, hyperstability. Relations between these various kinds of stability are discussed by Z. Moszner in [83], [90] (among others).

1.6 Ulam's problem with unbounded control function

Let me mention also the Ulam's problem with an unbounded control function[3]. Such research began at the turn of the 1940s and 1950s (see T. Aoki [3]). I will quote here a theorem from a book of N. J. Kalton, N. T. Peck and J. W. Roberts [61].

Theorem 1.7 (N. J. Kalton, N. T. Peck, J. W. Roberts). *A Banach space X is a K-space if and only if for every homogenuous function $f : X \to \mathbb{R}$ satisfying*

$$| f(x+y) - f(x) - f(y) | \leq \varepsilon(\|x\| + \|y\|), \quad x, y \in X$$

[2] The following theorem can be consider as an exponential counterpart of Hyers Theorem:

Theorem 1.6 (D. Kazhdan, [64]). *let G be a topological group which is amenable, and let $\mathcal{U}$ be a group of unitary operators on a Hilbert space H. Then, for every function $f : G \to \mathcal{U}$ satisfying*

$$\|f(x+y) - f(x)f(y)\| \leq \varepsilon, \quad x, y \in G$$

and $\varepsilon < \frac{1}{100}$ there exists a representation $\tau : G \to \mathcal{U}$ of group G such that

$$\|\tau(x) - f(x)\| \leq \varepsilon, \quad x \in G.$$

In general, Ulam's problem of stability of functional equation of exponential function (for maps with vector values) has not been solved until today. Some results can be found in paper of R. Ger, P. Šemrl [37] and in a monograph of D. H. Hyers, G. Isac i Th. M. Rassias [43], and further information about the quasirepresentations of groups was included in a survey of A. I. Shtern [100] and a paper of M. Burger, N. Ozawa, A. Thom [10].

[3] Namely, we replace ε from the right hand side of inequality (1.1) by a function depending on x and y.

with some ε, there exists a linear functional (not necessarily continuous) mapping L into X such that

$$| f(x) - L(x) | \leq K\|x\|,$$

for some constant K and all $x \in X$.

The aforementioned theorem turned out to be an inspiration for other authors investigating the problem of Ulam. Additionaly, this stability property proved to be very useful for mathematicians dealing with the K-space theory.

Other stability results with unbounded control function can be found in monograph [43], and further connections between K-space theory and stability theory, among others, in paper of F. Cabello Sánchez [11]. A paper of F. Cabello Sánchez and J. M. F. Castillo [12] is also worth mentioning, in which the relationship between the problem of Ulam and twisted sums of Banach spaces is examined. The multiplicative counterparts of the above theorem can be found in a book of K. Jarosz [53] and articles of B. E. Johnson [54], [55].

1.7 Ulam's problem for inequalities

We meet with yet another situation while considering Ulam's problem for inequalities. Already in 1950s classic Ulam's problem for convexity was considered by D. H. Hyers and S. M. Ulam [44].

Theorem 1.8 (D. H. Hyers, S. M. Ulam)**.** *If f is a real function defined on a convex subset D of $\mathbb{R}^n$ satisfying, with some $\varepsilon \geq 0$, inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon,$$

for all $x, y \in D$ and $t \in [0,1]$, then there exists a convex function $g : D \to \mathbb{R}$ such that

$$|f(x) - g(x)| \leq k_n\varepsilon, \quad x \in D,$$

where $k_n = (n^2 + 3n)/(4n + 4)$.

In this theorem, the dependence of stability constants on the dimension of space was obtained, and Z. Kominek and J. Mrowiec in [70] showed the lack of stability in the case of infinitely dimensional spaces. The problem of stability of convex function was considered later in 1984 by P. W. Cholewa in [17] (he improved the constants k_n), and at the end of the century by M. Laczkovich in [72]. In 2011 a paper of M. Laczkovich and R. Paulin [73] was published; the authors introduced another constant corresponding to bounded functions (second Whitney constant) and showed its connection to the stability constant of the Jensen equation.

1.8 Summary

Above, I presented only a brief and selective introduction to the theory of stability of functional equations. More of interesting results can be found in the monographs of D. H. Hyers, G. Isac and Th. M. Rassias "Stability of Functional Equations in Several Variables" [43] and in the survey paper of G. L. Forti [28], R. Ger [36] and L. Székelyhidi [109].

Despite the passing of many years, the problem of stability of functional equations is still alive and attracts the interest of many mathematicians.

1.9 My results in this field

My scientific achievement is part of the research line related to the problem of Ulam - the problem of stability of functional equations. When selecting publications for scientific achievement, not only did I want to present the results obtained, but also to show the examples of methods used to solve stability problems. I tried to list various problems arising from S. Ulam's question, and also pointed out the variety of methods that I used to achieve the results described below. Hence the conscious choice of papers seemingly distant from each other, but connected by a common denominator, which is S. Ulam's question from 1940.

Papers [A] and [C] refer to the stability problem of one of the most important functional equations - the translation equation. This problem put up many years ago is not fully solved until today. My results published in [A] give a positive answer in the class of continuous functions defined on $\mathbb{R} \times I$, where I is a real interval. Paper [C] is, in a sense, a refinement and complement to the paper [A]. In addition, it also raises the issue of "reverse stability" - we ask if the fact of being in close proximity to the exact solution results in being an approximate solution. In the case of the translation equation (in this situation) the answer is negative, but in [C] the conditions under which it will be so are given.

In papers [B], [E], [F] and [G] I remain faithful to the original problem of Ulam – the problem of stability of the Cauchy equation. Nevertheless, the methods presented in these articles are rather different from those used so far.

In the paper [B], a new technique of proof of Hyers theorem based on the Markov-Kakutani theorem is presented. The results contained in [D] show that this technique can be successfully used also in the stability study of other functional equations, using also other common fixed point theorems.

In publications [E], [F] Ulam's problem for the Cauchy's equation is combined with the problem of Erdős [22] and I. Farah's approach from the paper [24]. We prove that when a set of such points (x, y), for which the value $f(x+y)$ and the sum $f(x)+f(y)$ are far from each other by at least ε is of a small measure (but perhaps positive), then there is a homomorphism F such that the set of x's for which the distance between $f(x)$ and $F(x)$ are not close enough to each other, is of small measure.

In [G] we extend the areas of research related to the problem of Ulam. Namely, we formulate and investigate the problem of Ulam for the homomorphisms of lattices. We define in two ways what it means to be "the approximate homomorphism" of the lattices and, in these two cases, we prove the stability of the Cauchy equation (properly understood).

2. Stability of the translation equation

2.1 Introduction

A functional equation of the form

$$F(s+t,x) = F(t, F(s,x)), \quad s,t \in T,\, x \in X, \tag{2.1}$$

where a function F is defined on a set $T \times X$ with values in a set X and T is a set with binary operation $+$ is called the translation equation. We can interprete the set T as time, then $F(t,x)$ denotes the place of point x in time t. It is convenient to put $f^t := F(t,\cdot)$, then the translation equation takes the form

$$f^t \circ f^s = f^{s+t}, \quad s,t \in T. \tag{2.2}$$

The translation equation is one of the most important functional equations, it appears in a natural way in many topics[4]. It links the iteration theory with the functional equations. The solutions of the the translation equation with $T = \mathbb{R}$, or $T = (0,\infty)$ (it is convenient to consider the form (2.2)) are these functions $f\colon X \to X$, such that the discrete process $(f^n)_{n\in\mathbb{N}}$ generated by f has a continuous extension to the real time, or positive real time. A family of continuous maps $\{f^t;\, t \in \mathbb{R}\}$ or $\{f^t;\, t \in (0,\infty)\}$, satisfying (2.2) is called an iteration group or iteration semigroup, respectively. Moreover, if the initial condition $f^0 = \mathrm{id}$ is satisfied, the family $\{f^t;\, t \in \mathbb{R}\}$ is called a dynamical system[5].

Some interesting issues discussed in the connection with the translation equation are, among others,: the form of the solutions in different settings and under different assumptions (continuity, differentiablity, monotonicity), regularity of the solutions (when the measurability of iteration groups or semigroups implies their continuity), embedding (and near embedding[6]) into continuous iteration semigroups.

2.2 The stability of the translation equation in some class of functions

There are only a few articles on the stability of the translation equation.

[4] Quite a long list of such topics can be found in surveys of Z. Moszner [82] i [84].

[5] More about the stability of the dynamical systems can be found in the chapter 8 of this presentation.

[6] see chapter 10 of this presentation.

Let me start with disscusing the case of continuous functions defined on $\mathbb{R}\times I$, where I is a real interval, since this is the class of functions I was dealing with in my research.

Results concerning that case can be found in the papers [14], [A] and [C].

More precisely, J. Chudziak under the following assumptions:

$$\text{(H)}\qquad \begin{array}{c} I \text{ is an open real interval, } G\colon \mathbb{R}\times I\to I,\\ \text{the function } G(\cdot,x_0)\colon \mathbb{R}\to I \text{ is a continuous surjection for some } x_0\in I,\\ |G(t,G(s,x_0))-G(s+t,x_0)|\le\delta, \text{ for } s,t\in\mathbb{R} \text{ and a } \delta>0, \end{array}$$

shown in [14] how to define a homeomorphism $g\colon \mathbb{R}\to I$ such that

$$|G(t,x)-g(g^{-1}(x)+t)|\le 9\delta,\quad t\in\mathbb{R},\, x\in I,$$

i.e., he proved that for a continuous iteration group F given by

$$F(t,x)=g(g^{-1}(x)+t),\quad x\in I,\, t\in\mathbb{R}, \tag{2.3}$$

we have

$$|G(t,x)-F(t,x)|\le 9\delta,\quad x\in I,\, t\in\mathbb{R}.$$

In the article [A], I assume that I is a real interval, a function $G\colon \mathbb{R}\times I\to I$ is continuous with respect to each variable and the following inequality

$$|G(s,G(t,x))-G(t+s,x)|\le\delta,\quad s,t\in\mathbb{R},\, x\in I, \tag{2.4}$$

is satisfied for some $\delta>0$. It turns out that there is some family $\mathcal{U}$ of open disjoint subintervals of the interval $V:=G(\mathbb{R}\times I)$ for which the assumptions (H) are satisfied, even more, for an arbitrary point $x\in U\in\mathcal{U}$ the trajectory of x, i.e., $\mathbb{R}\ni t\mapsto G(t,x)\in I$, is a surjection onto U. This implies that on every interval $U\in\mathcal{U}$ we can define F by (2.3).

For every $x\in V\setminus\bigcup\mathcal{U}$ the interval $G(\mathbb{R}\times\{x\})$ has length no bigger than 8δ, so we can approximate a function $t\mapsto G(t,x)$ by the constant function $t\mapsto G(0,x)=:F(t,x)$.

For $x\notin V$ we define $F(t,x)$ according to the value of $G(0,x)$.

What is left is to take care of the continuity of F on the boundary of V, hence, some necessary modifications are needed. They result in the following formula for F:

$$F(t,x)=\begin{cases} g_\lambda(g_\lambda^{-1}(f(x))+t), & \text{if } f(x)\in U_\lambda,\ t\in\mathbb{R};\\ f(x), & \text{if } f(x)\notin\bigcup_{\lambda\in\Lambda}U_\lambda,\ t\in\mathbb{R}\end{cases}$$

where $\mathcal{U} = \{U_\lambda :\ \lambda \in \Lambda\}$ and

$$f(x) = \begin{cases} x, & \text{if } x \in [G(0,\inf V), G(0,\sup V)] \cap I, \\ G(0,\inf V), & \text{if } x \in [\inf V, G(0,\inf V)] \cap I, \\ G(0,\sup V), & \text{if } x \in [G(0,\sup V), \sup V] \cap I, \\ G(0,x), & \text{if } x \in I \setminus V \text{ and } G(0,x) \in [G(0,\inf V), G(0,\sup V)], \\ G(0,\inf V), & \text{if } x \in I \setminus V \text{ and } G(0,x) \in [\inf V, G(0,\inf V)], \\ G(0,\sup V), & \text{if } x \in I \setminus V \text{ and } G(0,x) \in [G(0,\sup V), \sup V]. \end{cases}$$

This was the sketch[7] of the proof of the stability of the translation equation from [A], i.e., the proof of the following theorem:

Theorem 2.1. *Let $I \subseteq \mathbb{R}$ be a real interval, $\delta \in (0,\infty)$, $G\colon \mathbb{R} \times I \to I$ be a function continuous with respect to each variable satisfying* (2.4). *Then there exists a continuous iteration group $F\colon \mathbb{R} \times I \to I$, such that*

$$|G(t,x) - F(t,x)| \leq 10\delta, \quad x \in I,\ t \in \mathbb{R}. \tag{2.5}$$

The reason to write the paper [C] was to emphasize some relations between F and G from the theorem above. In more details we list some facts which follow from (2.4) (see [C, Theorem 2.2]), moreover, we inverse this theorem, i.e., we indicate which conditions together with (2.5) guarantee satisfying (2.4) (see [C, Theorem 3.1]).

I am going to list these conditions, but first, let me start with reminding the commonly known[8] characterization of continuous solutions of the translation equation (where $I \subseteq \mathbb{R}$ is an interval and $T = \mathbb{R}$), to indicate some similarities between the form of H satisfying the translation equation approximately, and F being a solution of this equation.

Theorem 2.2. *Let $F\colon \mathbb{R} \times I \to I$ be a continuous solution to the translation equation, $V = F(\mathbb{R}\times I)$. Then there are open disjoint intervals $U_\lambda \subset V$, $\lambda \in \Lambda$, and homeomorphisms $h_\lambda\colon \mathbb{R} \to U_\lambda$, such that for every $x \in U_\lambda$ we have*

$$F(t,x) = h_\lambda(h_\lambda^{-1}(x) + t), \quad t \in \mathbb{R},$$

and

$$F(t,x) = x, \quad x \in V \setminus \bigcup_{\lambda\in\Lambda} U_\lambda,\ t \in \mathbb{R}.$$

[7]The full proof consists of some lemmas, corollaries from Section 2 of [A] and the "proper proof" from Section 3 of that article (pages 1982-1986).

[8]Unforunately, I cannot pinpoint where this characterization appeared for the first time; it can be found for example in the book of Z. Moszner [87], Chapter IX, 4D/ or it can be deduced from the monograph [101].

Moreover, there exists a continuous function $f\colon I\to V$*, such that* $f(x)=x$ *for* $x\in V$ *and*

$$F(t,x)=F(t,f(x)),\quad t\in\mathbb{R},x\in I\setminus V.$$

Conversely, for every continuous function $f\colon I\to I$*, such that* $f\circ f=f$*, a family of open disjoint intervals* $\{U_\lambda:\ \lambda\in\Lambda\}$ *such that* $U_\lambda\subset f(I)$ *and a family of homeomorphisms* $\{h_\lambda\colon\mathbb{R}\to U_\lambda:\ \lambda\in\Lambda\}$*, the function* F *given by*

$$F(t,x)=\begin{cases} h_\lambda(h_\lambda^{-1}(f(x))+t), & \text{if } f(x)\in U_\lambda,\ t\in\mathbb{R};\\ f(x), & \text{if } f(x)\notin\bigcup_{\lambda\in\Lambda}U_\lambda,\ t\in\mathbb{R}\end{cases}\tag{2.6}$$

$$\left(\text{i.e., by the formula } F(t,x)=\begin{cases} h_\lambda(h_\lambda^{-1}(x)+t), & \text{if }\ x\in U_\lambda,\ t\in\mathbb{R};\\ x, & \text{if }\ x\in f(I)\setminus\bigcup_{\lambda\in\Lambda}U_\lambda,\ t\in\mathbb{R};\\ F(t,f(x)), & \text{if } x\in I\setminus f(I)\end{cases}\right)$$

is a continuous solution to the translation equation.

Below, I list some conditions which follow from (2.4) ([C, Theorem 2.2]). Of course, each continuous iteration group F satisfies (2.4) with $\delta=0$, so, it satisfies the following conditions (a)–(m) with $\delta=0$.

Suppose that $G\colon\mathbb{R}\times I\to I$ is a continuous solution of the following inequality

$$|G(s,G(t,x))-G(s+t,x)|\le\delta,\quad x\in I,\ s,t\in\mathbb{R}.$$

Then:

(a) there exist families $\mathcal{U}=\{U_\lambda\subset I:\ \lambda\in\Lambda\}$ of open and disjoint intervals of the length greater or equal to 6δ and $\{h_\lambda\colon\mathbb{R}\to U_\lambda:\ \lambda\in\Lambda\}$ of homeomorphisms, and a continuous function $f\colon I\to I$, such that $f\circ f=f$, $\bigcup_{\lambda\in\Lambda}U_\lambda\subset f(I)$,

$$|G(t,x)-f(x)|\le 10\delta,\qquad t\in\mathbb{R},\ f(x)\notin\bigcup_{\lambda\in\Lambda}U_\lambda,$$

$$|G(t,x)-h_\lambda(h_\lambda^{-1}(f(x))+t)|\le 10\delta,\qquad t\in\mathbb{R},\ f(x)\in U_\lambda,\ \lambda\in\Lambda$$

(in particular, there exists a continuous solution F (given by (2.6)) to the translation equation, such that $|G-F|\le 10\delta$);

(b) $\forall_{(x\in I,\ U\in\mathcal{U})}\ (f(x)\in U\ \Rightarrow\ G(\mathbb{R}\times\{x\})=U)$

(if $f(x)\in U\in\mathcal{U}$, then the trajectory of x is a surjection onto U);

(c) $\forall_{(x\in I)}\ (x\in\bigcup\mathcal{U}\ \Rightarrow\ f(x)=x)$;

(d) $\forall_{(x\in I,\,t\in\mathbb{R})}\ (|f(G(t,x))-G(t,x)|\le 2\delta)$

(it means, for y belonging to the set of values of G, the value $f(y)$ is close to y);

(e) $\forall_{(x\in I)}\ (f(x)\notin\bigcup\mathcal{U}\ \Rightarrow\ (\forall_{t\in\mathbb{R}}\,f(G(t,x))\notin\bigcup\mathcal{U}))$;

(f) $\forall_{x\in I}\ (f(x)\notin\bigcup\mathcal{U}\ \Rightarrow\ (\forall_{s_1,s_2\in\mathbb{R}}\,|H(s_1,x)-H(s_2,x)|\le 6\delta))$

(trajectories of x, such that $f(x)\notin\bigcup\mathcal{U}$, are "short");

(g) the set of values of f is a subset of the set of values of G,

(h) every interval $U\in\mathcal{U}$ is "invariant", i.e.,

$$G(\mathbb{R}\times\{x\})=U,\qquad x\in U\in\mathcal{U},$$

and

$$G(\{t\}\times U)=U,\qquad t\in\mathbb{R},\,U\in\mathcal{U};$$

(i) either h_λ is an increasing homoemorphism, and then

$$\lim_{t\to\infty}G(t,x)=\sup U_\lambda=:b_\lambda,\quad \lim_{t\to-\infty}G(t,x)=\inf U_\lambda=:a_\lambda,\qquad x\in U_\lambda,$$

and for every $t\in\mathbb{R}$ we have $G(s,x)>G(t,x)-2\delta$ for $s>t$;
or h_λ is a decreasing homeomorphism, and then

$$\lim_{t\to\infty}G(t,x)=a_\lambda,\quad \lim_{t\to-\infty}G(t,x)=b_\lambda,\qquad x\in U_\lambda,$$

and for every $t\in\mathbb{R}$ we have $G(s,x)<G(t,x)+2\delta$ for $s>t$

(in this point I described the trajectories of $x\in U_\lambda$: if h_λ is an increasing homeomorphisms then $G(\cdot,x)$ is "almost" increasing, if h_λ is a decreasing homeomorphism then $G(\cdot,x)$ is "almost" decreasing');

(j) for every $\lambda\in\Lambda$, such that $a_\lambda\in I$:

$$G(t,a_\lambda)=a_\lambda,\qquad t\in\mathbb{R};$$

for every $\lambda\in\Lambda$, such that $b_\lambda\in I$:

$$G(t,b_\lambda)=b_\lambda,\qquad t\in\mathbb{R}$$

(the trajectories of the endpoints of U_λ are constant);

(k) for every $x\in I$ such that $x\notin\bigcup\limits_{\lambda\in\Lambda}U_\lambda$ but there exist $n,m\in\Lambda$, such that $b_n\le x\le a_m$, we have

$$|G(t,x)-x|\le 6\delta,\qquad t\in\mathbb{R}$$

(for x's which are between some two intervals from the family $\mathcal{U}$, the values $G(t,x)$ are close to x);

(l)

$$|G(t,x)-G(t,f(x))|\le 10\delta,\qquad t\in\mathbb{R},\,x\in I;$$

moreover

(m) for every $\lambda \in \Lambda$ we have two possibilities:

• either there exists an $\eta_\lambda > 0$, such that

$$|t_1 - t_2| \le \eta_\lambda \ \Rightarrow |h_\lambda(t_1) - h_\lambda(t_2)| \le 21\delta, \quad t_1, t_2 \in \mathbb{R}, \tag{2.7}$$

and then for $\eta_\lambda^* := \sup\{\eta_\lambda > 0 : \ (2.7)\} \in (0, \infty]$ we have

$$h_\lambda(t - \eta_\lambda^* + h_\lambda^{-1}(f(x))) \le G(t,x) \le h_\lambda(t + \eta_\lambda^* + h_\lambda^{-1}(f(x))), \quad t \in \mathbb{R},\ f(x) \in U_\lambda,$$

if h_λ increases ($h_\lambda(\pm\infty)$ denotes $\lim\limits_{t \to \pm\infty} h_\lambda(t)$) and

$$h_\lambda(t - \eta_\lambda^* + h_\lambda^{-1}(f(x))) \ge G(t,x) \ge h_\lambda(t + \eta_\lambda^* + h_\lambda^{-1}(f(x))), \quad t \in \mathbb{R},\ f(x) \in U_\lambda,$$

if h_λ decreases,

• or η_λ, such that (2.7) holds true does not exist and then

$$G(t,x) = h_\lambda(t + h_\lambda^{-1}(f(x)), \quad t \in \mathbb{R},\ f(x) \in U_\lambda$$

This point provides an estimation of the distance between $G(t,x)$ and $h_\lambda(t + h_\lambda^{-1}(f(x))$ for x such that $f(x) \in U_\lambda$ for some $\lambda \in \Lambda$, better than it was described in the point (a). Either this distance is exactly equal to zero (so the restriction of G to $\mathbb{R} \times U_\lambda$ is an exact solution of the translation equation, not only an approximate one), or we can at least control how fast h_λ increases, in such a case the equality between $G(t,x)$ and $h_\lambda(t + h_\lambda^{-1}(f(x))$ does not necessarily takes place, however, the inequalities form (m) shows how close $G(t,x)$ and the values $h_\lambda(t + h_\lambda^{-1}(f(x))$ are – in particular for t close to $\pm\infty$. From these inequalities we can deduce that

$$\lim_{t \to \pm\infty} |G(t,x) - h_\lambda(t + h_\lambda^{-1}(f(x))| = 0.$$

It is known[9], that the existence of the solution F of the translation equation in the vicinity of G, i.e., satisfying inequality (2.5), does not guarantee that G satisfies the translation equation approximately: (2.4). The theorem below ([C, Theorem 3.1]) provides the conditions which together with (2.5) guarantee that (2.4) holds true.

Theorem 2.3. *Let I be a nondegenerate real interval, δ, A_1, A_2, B, C, $D > 0$, and let $H\colon \mathbb{R} \times I \to I$ be a continuous function. Suppose that*

(a') *there exist open disjoint intervals $U_n \subset I$, $n \in N$, with $N \subset \mathbb{N}$ being some set of indices, homeomorphisms $h_n\colon \mathbb{R} \to U_n$, $n \in N$, and a continuous function $f\colon I \to I$ such that $f \circ f = f$, $U_n \subset f(I)$, $n \in N$,*

$$|H(t,x) - f(x)| \le A_1\delta, \qquad t \in \mathbb{R},\ f(x) \notin \bigcup_{n \in N} U_n,$$

[9] see Theorem 4.4 i Theorem 4.5 from [X].

$$|H(t,x) - h_n(h_n^{-1}(f(x)) + t)| \le A_2\delta, \qquad t \in \mathbb{R},\ f(x) \in U_n,\ n \in N;$$

(b') $\forall_{(x\in I,\, n\in N)}\ (f(x) \in U_n \ \Rightarrow\ H(\mathbb{R},x) \subset U_n)$;

(c') $\forall_{(x\in I,\, n\in N)}\ (x \in U_n \ \Rightarrow\ f(x) = x)$;

(d') $\forall_{(x\in I,\, t\in\mathbb{R})}\ (|f(H(t,x)) - H(t,x)| \le B\delta)$;

(e') $\forall_{x\in I}\left(f(x) \notin \bigcup_{n\in N} U_n \ \Rightarrow\ \left(\forall_{t\in\mathbb{R}}\, f(H(t,x)) \notin \bigcup_{n\in N} U_n\right)\right)$;

(f') $\forall_{x\in I}\left(f(x) \notin \bigcup_{n\in N} U_n \ \Rightarrow\ (\forall_{s_1,s_2\in\mathbb{R}}\, |H(s_1,x) - H(s_2,x)| \le C\delta)\right)$;

moreover

(m') *for every $n \in N$ one of the following two possibilities holds:*

- *either there exists an $\eta_n > 0$, such that*

$$|t_1 - t_2| \le \eta_n \ \Rightarrow |h_n(t_1) - h_n(t_2)| \le D\delta, \qquad t_1, t_2 \in \mathbb{R}, \tag{2.8}$$

and for $\eta_n^* := \sup\{\eta_n > 0 :\ (2.8)\} \in (0,\infty]$ we have

$$h_n(t - \eta_n^* + h_n^{-1}(f(x))) \le H(t,x) \le h_n(t + \eta_n^* + h_n^{-1}(f(x))), \qquad t \in \mathbb{R},\ f(x) \in U_n,$$

if h_n increases, or

$$h_n(t - \eta_n^* + h_n^{-1}(f(x))) \ge H(t,x) \ge h_n(t + \eta_n^* + h_n^{-1}(f(x))), \qquad t \in \mathbb{R},\ f(x) \in U_n,$$

if h_n decreases,

- *or such an η_n, for which (2.8) holds true does not exist and then*

$$H(t,x) = h_n(t + h_n^{-1}(f(x)), \qquad t \in \mathbb{R},\ f(x) \in U_n.$$

Then

$$|H(s, H(t,x)) - H(t+s,x)| \le E\delta, \qquad s,t \in \mathbb{R},\ x \in I,$$

where $E := \max\{(2A_2 + D), \min\{3A_1 + B, A_1 + B + C\}\}$.

The second aim of writing [A] was to investigate the stability of the translation equation in the class of surjections and to investigate the stability of the system of equations

$$\begin{cases} H(s, H(t,x)) = H(t+s,x), \\ H(0,x) = x. \end{cases}$$

It turned out that the translation equation is stable in the class of surjective function but the above system of equations is not stable for $I \subsetneq \mathbb{R}$. However, a function H satisfying the translation equation is surjective if and only if it satisfies the identity condition $H(0,x) = x$. It gave the motivation to start the cowork with Prof. Zenon Moszner concerning the stablity (in different senses) of dynamical systems (in different senses, too). The results of this cooperation include [X] (later this research was continued by Z. Moszner in [89] i [90]).

2.3 The other results concerning Ulam's problem for the translation equation

In [77] there was investigated the b-stability[10] and stability in the sense of Hyers-Ulam[11] of equation (2.1). This article contains a few remarks[12] concerning some special cases[13]. The

[10]Let $(T,+)$ be a monoid with a neutral element, (X,ρ) a metric space. We say that the equation (2.1) is b-stable if for every $G\colon T\times X\to X$ the following implication holds: if

$$\{\rho(G(s,G(t,x)),G(t+s,x)):\, x\in X,\ s,t\in T\}$$

is bounded then there exists a solution F of (2.1), such that the set

$$\{\rho(G(s,x),F(s,x)):\, x\in X,\ s\in T\}$$

is bounded.

[11]We say that the equation (2.1) is jest *stable in the Hyers-Ulam sense* , if for every $\varepsilon>0$ there exists a $\delta>0$, such that for every $G\colon T\times X\to X$ satisfying

$$\rho(G(s,G(t,x)),G(t+s,x))\le\delta,\quad x\in X,\ s,t\in T, \tag{2.9}$$

there exists a solution F to (2.1), such that

$$\rho(G(s,x),F(s,x))\le\varepsilon,\quad x\in X,\ s\in T. \tag{2.10}$$

[12]First three come from an earlier article [85].

[13]These are:

- if T is a free group generated by 2 elements, X is the set of integers with natural metric then the equation (2.1) is not b-stable;
- if T is a groupoid, X is the set of integers with natural metric then the equation (2.1) is Hyers-Ulam stable;
- if $T=\{0\}$ is the trivial group, X an arbitrary metric space, then the equation (2.1) is both b-stable and Hyers-Ulam stable;
- if $T=\{0,1\}$ is the group of two elements, $X=\{0,1,1/2,1/3,\ldots\}$ with natural metric, then there is not true that:

for every $\varepsilon>0$ there exists a $\delta>0$, such that for every $G\colon T\times X\to X$ satisfying (2.9) there exists a solution F of an equation (2.1), such that

$$F(0,x)=x,\quad x\in X$$

and (2.10) holds true.

main results[14] from the article [77] concern the stability of the equation (2.1) in classes

$$\mathcal{B} = \{H\colon T \times X \to X : \ H(\cdot, x_0) \text{ is bijection for some } x_0 \in X\}$$

and

$$\begin{aligned}\mathcal{I} = \{H\colon T \times X \to X : \ \ & H(\cdot, x_0) \text{ is injection} \\ & \text{and } H(T \times \{x_0\}) = H(\{0\} \times X) \ \text{ for some } x_0 \in X\}\end{aligned}$$

Authors of the paper [48] considered the stability of the translation equation in the ring of formal power series[15] $\mathbb{K}[X]$ over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. They proved in [48] that the

[14] In [77] there was proved that, if $G \in \mathcal{B}$ and $g := G(\cdot, x_0)\colon T \to X$ is a bijection, then the function $F\colon T \times X \to X$ given by $F(t, x) = g(g^{-1}(x) + t)$ is a solution to the translation equation and belongs to the class $\mathcal{B}$ (more precisely, $F(\cdot, g(0))$ is a bijection). Moreover,

$$G(t, x) = G(t, G(g^{-1}(x), x_0)), \quad t \in T, x \in X,$$

and

$$F(t, x) = G(g^{-1}(x) + t, x_0).$$

Hence, the equation (2.1) is obviously stable. Furthermore, if $G \in \mathcal{I}$, $g := G(\cdot, x_0)\colon T \to X$ is injection and $G(T \times \{x_0\}) = G(\{0\} \times X) =: X_0$, then the function $F\colon T \times X \to X$ given by

$$F(t, x) = g(g^{-1}(f(x)) + t),$$

where

$$f(x) = \begin{cases} x, & \text{for } x \in X_0, \\ G(0, x), & \text{for } x \in X \setminus X_0, \end{cases}$$

is a solution to the translation equation and belongs to the class $\mathcal{I}$. Implication (2.9) $\Rightarrow$ (2.10) holds with $\epsilon = 2\delta$.

[15] There $F(t, X)$ is a formal power series, i.e., it is of the form $\sum_{i=1}^{\infty} c_i(t)X^i$ for some coefficients $c_i\colon G \to \mathbb{K}$, moreover by $F(t, F(s, X))$ we understand the series obtained by substitution $\sum_{i=1}^{\infty} c_i(t)(F(s, X))^i$. For a series $p(X) = \sum_{i=0}^{\infty} c_i X^i$ we define $\operatorname{ord}(p(X))$ as $\min\{i \in \mathbb{N} \cup \{0\} : c_i \neq 0\}$.

translation equation is stable[16] under some assumption on a group G (see [48, Theorem 2 i Theorem 3]).

3. Common fixed point theorems in the theory of stability of functional equations

3.1 Introduction

In papers [B] and [D] I have developed a method of proving the stability of some functional equation. This method relies on common fixed point theorems, more precisely, in papers [B] and [D] I used the following theorems:

Theorem 3.1 (A. Markow [79], S. Kakutani [59], [99]). *Let $\mathcal{X}$ be a linear-topological space, $\mathcal{K} \subset \mathcal{X}$ be a nonempty convex compact subset of $\mathcal{X}$. Assume that $\mathcal{F}$ is a family of continuous affine selfmaps of $\mathcal{K}$ such that*

$$F \circ G = G \circ F, \quad F, G \in \mathcal{F}.$$

Then there exists a $y \in \mathcal{K}$, such that $F(y) = y$ for every $F \in \mathcal{F}$.

Theorem 3.2 (R. DeMarr [20], [76]). *Let $(C, \leq)$ be a complete partially ordered set with the largest element. Suppose that $\mathcal{F}$ is a commuting[17] family of nondecreasing selfmaps of C. Then there exists a common fixed point of all selfmaps from $\mathcal{F}$ in the set C.*

---

[16] We say that the translation equation $F(t, F(s, X)) = F(s + t, X)$ is stable if for every positive integer N there is a positive integer M, such that for every family $(F(t, X))_{t \in G}$ of formal power series

$$F(t, X) = \sum_{i=1}^{\infty} c_i(t) X^i, \quad t \in G,$$

if

$$\operatorname{ord}(F(t + s, X) - F(s, F(t, X))) > M, \quad s, t \in G,$$

then there exists a group $(\bar{F}(t, X))_{t \in G}$ of formal power series

$$\bar{F}(t, X) = \sum_{i=1}^{\infty} \bar{c}_i X^i, \quad t \in G,$$

such that

$$\operatorname{ord}(F(t, X) - \bar{F}(t, X)) > N, \quad t \in G,$$

(i.e., $c_i = \bar{c}_i$ for $1 \leq i \leq N$).

[17] i.e.,

$$F \circ G = G \circ F \quad \text{dla} \quad F, G \in \mathcal{F}$$

.

3.2 A new proof of the Hyers Theorem

Below, I present a sketch of a new proof of the following version of the Hyers Theorem[18]; a proof from the paper [B]:

Theorem 3.3 (Hyers, [42])**.** *Let* $(S,+)$ *be an abelian semigroup,* $\varepsilon \geq 0$*,* $\varphi\colon S \to \mathbb{K}$*, where* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$*. Suppose that*

$$|\varphi(x+y) - \varphi(x) - \varphi(y)| \leq \varepsilon, \quad x, y \in S.$$

Then there exists an additive function $a\colon S \to \mathbb{K}$ *such that*

$$|a(s) - \varphi(s)| \leq \varepsilon, \quad s \in S.$$

Sketch of the proof. Put $\mathcal{X} = \ell_\infty(S)$, i.e., the space of all bounded functions defined on S with the values in $\mathbb{K}$ endowed with the supremum norm. Let $\ell_1(S)$ be the space of all summable functions defined on the set S with values in $\mathbb{K}$ endowed with the norm $\|f\| = \sum_{s\in S} |f(s)|$. Since $\mathcal{X} = (\ell_1(S))^*$, we can consider the space $\mathcal{X}$ with weak$^*-$topology and with this topology it is a linear-topological space. A family $\mathcal{F} = \{T_x\colon \mathcal{X} \to \mathcal{X};\, x \in S\}$, where

$$T_x(f) := f(x+\cdot) + \varphi(x+\cdot) - f(x) - \varphi(x) - \varphi(\cdot), \quad x \in S,$$

is a commuting family of continuous affine selfmaps of $\mathcal{X}$. Moreover the set

$$\mathcal{C} := \{f \in Y : \|f\| \leq \varepsilon,\, \|T_x(f)\| \leq \varepsilon,\, x \in S\}$$

is nonempty convex and $T_x(\mathcal{C}) \subset \mathcal{C}$ for every $x \in S$. Its weak*-closure, $\mathcal{K}$, is a nonempty weak*-compact convex and invariant for every T_x, where $x \in S$. From Markov-Kakutani Theorem we infer that there exists an $f \in \mathcal{K}$, such that $T_x(f) = f$ for all $x \in S$. Putting $a := f + \varphi$ we get that $a\colon S \to \mathbb{K}$ is additive and $\|a - \varphi\| = \|f\| \leq \varepsilon$. $\square$

The reasoning presented above shows another method of proving Theorem 3.3 (after the so called "direct method" [42], the method relying on some Banach contraction principle -type theorem [96], [13] and the invariant mean method introduced in [106]).

3.3 An application of the common fixed point theorems to the stability of some other functional equations

The aim of the paper [D] was to show the usage of the common fixed point theorems in proving the stability of the functional equation of the form

$$f(s \diamond x) = F(s, f(x)), \quad s \in G, x \in X, \tag{3.1}$$

[18] Let me remind that the original Hyers Theorem concerned approximate homomorphisms between two Banach spaces. Let me remind also Theorem 1.3 for justifying why the target space in Theorem 3.3 is "only" $\mathbb{R}$ or $\mathbb{C}$.

where G is an abelian group acting on a set X, Y is a set, $F\colon G\times Y\to Y$ is a given function and an unknown function f is defined on the set X and takes its values in Y. Considering this equation was inspired by a functional equation from [1]:

$$f(sx)=F(s,f(x)),\quad s\in S, x\in X, \tag{3.2}$$

with unknown function $f\colon S\to S$, where S is a semigroup with a neutral element, $F\colon S\times S\to S$ is a given function. It is worth noticing that some particular cases of (3.1) are

- the homogenity equation:

$$f(sx)=s^p f(x),\quad s\in\mathbb{K}_0,\ x\in X, \tag{3.3}$$

  where $\mathbb{K}_0$ is a subgroup of the group $(\mathbb{R}\setminus\{0\},\cdot)$ or $(\mathbb{C}\setminus\{0\},\cdot)$, and X, Y are a linear spaces over $\mathbb{R}$ or $\mathbb{C}$, respectively;
- the periodic function equation:

$$f(x+kp)=f(x),\quad x\in X,\ k\in\mathbb{Z}, \tag{3.4}$$

  where X is a group and p is its certain element;
- the microperiodic function equation:

$$f(x+qp)=f(x),\quad x\in X,\ q\in\mathbb{Q}, \tag{3.5}$$

  where X is a linear space and p is its certain element;
- equation

$$f(x^y)=yf(x),\quad x\in(0,\infty),\ y\in\mathbb{R}\setminus\{0\}; \tag{3.6}$$

- equation

$$f(s+t)=s+f(t),\quad s\in G_0,\ t\in G, \tag{3.7}$$

  where G_0 is a subgroup of a group G;
- equation

$$f(x^p)=f(x)^p,\quad x\in(0,\infty),\ p\in G, \tag{3.8}$$

  where G is a subgroup of $((0,\infty),\cdot)$.

The stability of the equations (3.3), (3.6) and (3.7), so, of the special cases of the equation (3.1), was investigated already in the papers [45, 46, 47, 49, 56, 69, 108, 111, 116]. Actually, even more was shown there, that the equations (3.3) and (3.6) are superstable, i.e., that every "approximate solution" is an "exact solution". In the paper [D] using the common fixed point method I have obtained "only" the stability of these equation, however, in [D, Corollary 5.1] I showed how to easily get their superstability from their stability.

Theorem 3.4 ([D], Theorem 3.2)**.** *Suppose that*

(i) G *is an abelian group acting on a set* X, Y *is linear-topological space;*

(ii) $F\colon G \times Y \to Y$ *satisfies the translation equation:*

$$F(s, F(t,x)) = F(st, x), \quad s,t \in G,\ y \in Y;$$

(iii) $F(t,\cdot)\colon Y \to Y$ *are continuous and affine for every* $t \in G$*;*

(iv) $K \subset Y$ *is a compact convex set such that* $0 \in K$, $f_0\colon X \to Y$ *and*

$$F(s, f_0(x)) - f_0(s \diamond x) \in K, \quad x \in X,\ s \in G.$$

Then there exists a solution $f\colon X \to Y$ *to the equation* (3.1)*, such that*

$$f(x) - f_0(x) \in K, \quad x \in X. \tag{3.9}$$

The assumption (ii) may seem quite strong, however, it is satisfied in each of the above mentioned special cases of the equation (3.1); moreover, as it was shown in [1], satisfying the translation equation at least in one point is a necessary condition for existence of solutions to the equation (3.2); furthermore, the assumption that the translation equation is satisfied at least in one point may be not sufficient to obtain stability ([D, Example 5.2]). The assumption (iii) is satisfied in examples (3.3)–(3.7). The assumption (iv) means that $f_0\colon X \to Y$ satisfies the equation (3.1) approximately. On the other hand (3.9) expresses that f is close to f_0.

Sketch of the proof of Theorem 3.4. We consider $\mathcal{X} := Y^X$ with the product topology and define $G_t\colon \mathcal{X} \to \mathcal{X}$, for $t \in G$, by the formula

$$G_t(f)(x) = F(t, f(t^{-1} \diamond x)), \quad f \in \mathcal{X},\ x \in X.$$

Let the set $\mathcal{K}$ consist of those function $f \in \mathcal{X}$, for which $f(x) - f_0(x) \in K$ for $x \in X$ and $G_t(f)(x) - f_0(x) \in K$ for any $x \in X$ i $t \in G$. It is enough to check that the assumptions of Theorem 3.1 (with $\mathcal{F} := \{G_t,\ t \in G\}$) are satisfied to get the existence of $f \in \mathcal{K}$ (which implies (3.9)) such that $G_t(f) = f$ for every $t \in G$ (this means that f is a solution of (3.1)). $\square$

In [D, Theorem 4.2] I assume that:

– Y is a poset;

– $f_0\colon X \to Y$ satisfies (3.1) approximately, more precisely, there exist $a, b\colon X \to Y$, such that $a(x) \le b(x)$ and

$$f_0(x), F(t, f_0(t^{-1} \diamond x)) \in [a(x), b(x)], \quad x \in X,\ t \in G,$$

– $F\colon G\times Y\to Y$ satisfies the translation equation, is increasing and "continuous" with respect to the second coordinate[19] (these assumptions about F allow us to use Theorem 3.2; let me emphasize that all of them are satisfied for (3.3)–(3.8) with suitable X, Y, G).

Sketch of the proof of stability the equation (3.1)*, i.e., of* [D, Theorem 4.2]. We consider $Z=Y^X$ with a partial order introduced by

$$f\le g \quad\Leftrightarrow\quad f(x)\le g(x) \ \text{ for } \ x\in X.$$

We define $G_t\colon Z\to Z$ for $t\in G$ by the formula

$$G_t(f)(x)=F(t,f(t^{-1}\diamond x)).$$

For a family

$$\mathcal{F}=\{G_t,\, t\in G\}$$

and the set

$$\mathcal{C}=\{f\in Z:\ f(x),G_t(f)(x)\in[a(x),b(x)]\text{ for } \ x\in X,\, t\in G\}$$

all the assumptions of Theorem 3.2 are satisfied, hence there exists in $\mathcal{C}$ a common fixed point f of the family $\mathcal{F}$. Since $G_t(f)=f$ for every $t\in G$ and $f\in\mathcal{C}$ we infer that
– f is a solution to (3.1),
– f is close to f_0 in the following sense:

$$f_0(x),f(x)\in[a(x),b(x)],\quad x\in X,$$

which means that the equation (3.1) is stable. □

4. Ulam's type problem for lattice homomorphisms

4.1 Introduction

A function $f\colon X\to Y$, where X and Y are lattices, is called:
$\vee$-*homomorphism*, if

$$f(x\vee y)=f(x)\vee f(y),\quad x,y\in X;$$

$\wedge$-*homomorphism*, if

$$f(x\wedge y)=f(x)\wedge f(y),\quad x,y\in X;$$

homomorphism, if it is both $\vee$-homomorphism and $\wedge$-homomorphism. Up to now there are only a few papers concerning the problem of stability (for homomorphisms) in lattices. Let me remind the paper of N. J. Kalton and J. W. Roberts with the following deep result:

[19]see [D, Theorem 4.2, assumptions (B)].

Theorem 4.1 (N. J. Kalton & J. W. Roberts [60]). *Let X be a boolean algebra, $f\colon X \to \mathbb{R}$ be a function satisfying the inequality*

$$|f(x \vee y) - f(x) - f(y)| \leq 1 \quad \text{for } x, y \in X \text{ such that } x \wedge y = 0.$$

Then there exists a map $g\colon X \to \mathbb{R}$, such that

$$g(x \vee y) = g(x) + g(y) \qquad \text{for } x, y \in X \text{ such that } x \wedge y = 0$$

and $|f(x) - g(x)| < 45$ for every $x \in X$.

This result is of fundamental importance in functional analysis, especially in theory of twisted sums of quasi-Banach spaces (see [62]), as well as in the stability problem for vector meaures (see [66]).

A somehow related result, very combinatorial in its nature, was obtained by I. Farah.

Theorem 4.2 (I. Farah [23]). *Let $n, m \in \mathbb{N}$, $X = 2^{\{1,2,\ldots,m\}}$ and $Y = 2^{\{1,2,\ldots,n\}}$. Suppose that $\varphi\colon Y \to [0, \infty]$ is a submeasure, i.e., $\varphi(\emptyset) = 0$, $\varphi(A) \leq \varphi(A \cup B)$, for $A, B \subset Y$ and $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$, for $A, B \subset Y$. Moreover, let us suppose that φ is nonpathological, i.e., it is supremum of all measures it dominates. Let $\varepsilon > 0$ and $f\colon X \to Y$ satisfy*

$$\varphi(f(x \cup y) \div (f(x) \cup f(y)) < \varepsilon \quad \text{for } x, y \in X,$$

$$\varphi(f(X \setminus x) \div (Y \setminus f(x))) < \varepsilon \quad \text{for } x \in X.$$

Then there exists a lattice homomorphism $g\colon X \to Y$, such that $\varphi(f(x) \div g(x)) < 521\varepsilon$ for every $x \in X$.

4.2 A description of the results from [G]

In [G] we proposed two ways of expressing that $f\colon X \to Y$ is an approximate $\vee$-homomorphism of lattices X i Y. The first uses the so called control function, the second way uses the system of neighbourhoods.

The proofs of two main results from [G] rely on the following separation lemma[20].

[20] In the paper [G] we indicate that some corollary can be deduced from Lemma 4.1 and the following theorem:

Theorem 4.3 (W. Kubiś [71]). *Let L be a distributive lattice, $\mathbb{B}$ be a complete Boolean algebra, $f, g\colon L \to \mathbb{B}$ and assume that f is a $\wedge$-homomorphism, g is a $\vee$-homomorphism and $f(x) \leq g(x)$ for $x \in L$. Then there exists a lattice homomorphism $h\colon L \to \mathbb{B}$ such that $f(x) \leq h(x) \leq g(x)$ for every $x \in L$.*

The mentioned corollary is written just before Theorem 7 from [G].

Lemma 4.1. *Let X be a distributive lattice and Y be a conditionally complete*[21] *lattice. Assume that maps $\Phi, \Psi \colon X \to Y$ satisfy the following conditions: $\Phi \le \Psi$,*

$$\Phi(x \vee y) \le \Phi(x) \vee \Phi(y) \quad x, y \in X,$$

and

$$\Psi(x \vee y) \ge \Psi(x) \vee \Psi(y) \quad x, y \in X.$$

Then there exists a $\vee$-homomorphism $F \colon X \to Y$, separating Ψ and Φ, it means such that $\Phi \le F \le \Psi$.

The proof is constructive.

Considering Ulam's problem with the constant control function turns out to be trivial, that's why we present in [G] two other possible approaches to this problem. The first one uses the so called control functions (see (4.1) below, to express the fact that f is an approximate $\vee$-homomorphism and (4.2), which indicates that f and F are close to each other).

Theorem 4.4. *Let X and Y be distributive lattices and assume that Y is conditionally complete and satisfies the dual to the infinite distributive law, that is,*

$$y \vee \inf S = \inf\{y \vee s \colon s \in S\},$$

for all $y \in Y$ and nonempty $S \subset Y$ bounded from below. Assume that maps $f \colon X \to Y$ and $\phi, \psi \colon X \times X \to Y$ satisfy the following conditions:

$$\phi(z, z) \le \phi(x, y) \quad \text{for } x, y, z \in X \text{ such that } x, y \le z,$$

$$\psi(x, y) \le \psi(z, z) \quad \text{for } x, y, z \in X \text{ such that } x, y \le z$$

and

$$\phi(x, y) \wedge f(x \vee y) \le f(x) \vee f(y) \le f(x \vee y) \vee \psi(x, y) \quad \text{for } x, y \in X. \tag{4.1}$$

Then there exists a $\vee$-homomorphism $F \colon X \to Y$, such that

$$\phi(x, x) \wedge f(x) \le F(x) \le f(x) \vee \psi(x, x) \quad \text{for } x \in X. \tag{4.2}$$

This proof is also constructive. First we define Φ and Ψ by

$$\Phi(x) = \inf\big\{f(x_1) \vee \ldots \vee f(x_n) \colon n \in \mathbb{N},\, x_1, \ldots, x_n \in X,\, x = x_1 \vee \ldots \vee x_n\big\}$$

and

$$\Psi(x) = \sup\big\{f(x_1) \vee \ldots \vee f(x_n) \colon n \in \mathbb{N},\, x_1, \ldots, x_n \in X,\, x = x_1 \vee \ldots \vee x_n\big\},$$

[21] i.e., every nonempty bounded subset has the largest and the smallest element.

and then we show that

$$\phi(x,x) \wedge f(x) \le \Phi(x) \le f(x), \quad x \in X,$$

$$f(x) \le \Psi(x) \le f(x) \vee \psi(x,x), \quad x \in X.$$

Moreover we notice that

$$\Phi(x \vee y) \le \Phi(x) \vee \Phi(y), \quad x, y \in X,$$

and

$$\Psi(x \vee y) \ge \Psi(x) \vee \Psi(y), \quad x, y \in X.$$

It enables us to make use of Lemma 4.1: the function F defined by

$$F(x) = \sup\{\Phi(z);\, z \le x\}, \quad x \in X.$$

satisfies all the desired conditions.

The second of our approaches to the Ulam's problem in lattices uses the system of neighbourhoods (see (4.3) and (4.4)).

Theorem 4.5. *Let X and Y be distributive lattices and assume that Y is conditionally complete and satisfies the dual to the infinite distributive law, that is,*

$$y \vee \inf S = \inf\{y \vee s \colon s \in S\}$$

f or all $y \in Y$ and nonempty $S \subset Y$ bounded from below. Assume moreover that there is a function $\mathcal{N} \colon Y \to 2^Y$ each of whose value is a bounded set, and which satisfies the following conditions:

(i) *$y \in \mathcal{N}(y)$ for every $y \in Y$;*
(ii) *if $t, u \in \mathcal{N}(z)$ and $t \le y \le u$, then $y \in \mathcal{N}(z)$;*
(iii) *$\sup \mathcal{N}(y) \in \mathcal{N}(y)$ and $\inf \mathcal{N}(y) \in \mathcal{N}(y)$ for every $y \in Y$;*
(iv) *if $t \in \mathcal{N}(u)$ and $u \vee y \in \mathcal{N}(z)$, then $t \vee y \in \mathcal{N}(z)$.*

Then for every map $f \colon X \to Y$ satisfying

$$f(x) \vee f(y) \in \mathcal{N}(f(x \vee y)) \quad \textit{for } x, y \in X \tag{4.3}$$

there exists a $\vee$-homomorphism $F \colon X \to Y$ such that

$$F(x) \in \mathcal{N}(f(x)), \quad \textit{for every } x \in X. \tag{4.4}$$

We gave a few natural examples of $\mathcal{N}$ satisfying conditions (i)–(iv).

Notice that in both theorems we can change $\vee$ into $\wedge$ to get the analogous results for approximate $\wedge$-homomorphisms.

In order to compare the above results with the already known theorems we considered the stability of monotonic functions.

Notice that $f\colon D \subset \mathbb{R} \to \mathbb{R}$ is
– increasing if and only if $\max\{f(x), f(y)\} = f(\max\{x, y\})$ for every $x, y \in D$,
– decreasing if and only if $\max\{f(x), f(y)\} = f(\min\{x, y\})$ for every $x, y \in D$.

Corollary 4.1. *Let $D \subset \mathbb{R}$, $\varepsilon \geq 0$, and assume that a function $f\colon D \to \mathbb{R}$.*

(a) *If*

$$\max\{f(x), f(y)\} - f(\max\{x, y\}) \leq \varepsilon \quad \text{for } x, y \in D,$$

then there exists an increasing function $g\colon D \to \mathbb{R}$, such that

$$|f(x) - g(x)| \leq \varepsilon/2, \qquad \text{for every } x \in D.$$

(b) *If*

$$\max\{f(x), f(y)\} - f(\min\{x, y\}) \leq \varepsilon \quad \text{for } x, y \in D,$$

then there exists a decreasing function $tg\colon D \to \mathbb{R}$ such that

$$|f(x) - g(x)| \leq \varepsilon/2, \qquad \text{for every } x \in D.$$

From this corollary we can deduce the following result[22].

Theorem 4.6 (W. Förg-Rob, K. Nikodem, Zs. Páles [30])**.** *Let $I \subset \mathbb{R}$ be an interval, $\varepsilon \geq 0$ and assume that a function $f\colon I \to \mathbb{R}$ satisfies*

$$\min\{f(x), f(y)\} - \varepsilon \leq f(tx + (1-t)y) \leq \max\{f(x), f(y)\} + \varepsilon$$

for $x, y \in I$, $t \in [0,1]$. Then there exists a monotone function $g\colon I \to \mathbb{R}$ such that

$$|f(x) - g(x)| \leq \varepsilon/2, \qquad \text{for every } x \in I.$$

Finally, we observed[23] that we can generalize the Corollary 4.1 to the following result:

Corollary 4.2. *Let D and E be linearly ordered sets and assume that E is conditionally complete. Let $\{I_\lambda : \lambda \in \Lambda\}$ be a fixed cover of E, where each I_λ is an interval. Assume that a function $f\colon D \to E$ satisfies the condition: for all $x, y \in D$ there is $\lambda \in \Lambda$ such that*

$$\{f(x \vee y), f(x) \vee f(y)\} \subset I_\lambda.$$

Assume also that for each $x \in D$ the set

$$I(x) := \bigcup\{I_\lambda : x \in I_\lambda\}$$

[22] But (when D is an interval) the Corollary 4.1 can be deduced from Theorem 4.6.
[23] thanks to the referee of [G].

is bounded above. Then, there exists an increasing function $F\colon D \to E$ *such that* $f(x) \leq F(x) \leq \sup I(x)$ *for every* $x \in D$.

5. Ulam's problem in relation to measure

5.1 Introduction

P. Erdős in [22] posed the following problem:

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+y) = f(x) + f(y) \quad \text{for } (x,y) \in (\mathbb{R} \times \mathbb{R}) \setminus Z,$$

where $Z \subset \mathbb{R} \times \mathbb{R}$ is of Lebesgue measure zero. Does there exist a function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$g(x+y) = g(x) + g(y) \quad \text{for all } (x,y) \in \mathbb{R} \times \mathbb{R},$$

and

$$g(x) = f(x) \quad \text{for } x \in \mathbb{R} \setminus U,$$

where $U \subset \mathbb{R}$ is of Lebesgue measure zero?

The positive answer to this problem can be found in papers of W. B. Jurkat [57] and N. G. de Bruijn [9]; see also R. Ger [32] and Ja. Tabor [110].

R. Ger in [33] connected Ulam's problem with the question of Erdős and proved (under some assumptions about groups G, H and σ-ideals in G and G^2) that if

$$d(f(x+y), f(x)+f(y)) \leq \delta, \quad \text{for "almost all" } (x,y) \in G^2,$$

then there exists an additive function $g\colon G \to H$ such that

$$d(f(x), g(x)) \leq \delta, \quad \text{for "almost all" } x \in G.$$

Similar problem was considered in the paper of I. Farah [24], however, the author indicates that his motivation was of a different nature. Let G and H be groups and μ a probability measure in G such that

$$\mu(a+X) = \mu(X), \;\; \mu(X+a) = \mu(X), \;\; \text{and} \;\; \mu(\{-x : x \in X\}) = \mu(X),$$

for measurable subsets X of group G and $a \in G$. For $\delta > 0$ he called a map $f : G \to H$ a δ-approximate homomorphism of type I with respect to μ if

$$\mu^2(\{(x,y) \in G \times G : \; f(x)+f(y) \neq f(x+y)\}) \leq \delta \tag{5.1}$$

and

$$\mu(\{x \in G : \; f(x) \neq -f(-x)\}) \leq \delta. \tag{5.2}$$

He presented the following theorem:

Theorem 5.1 (I. Farah, [24])**.** *If G is finite, μ is the uniform probability measure*[24] *on G, $f : G \to H$ is a δ-approximate homomorphism of type I with respect to μ and $\delta \leq \frac{1}{11}$, then there is a homomorphism $h : G \to H$ such that*

$$\mu(\{x : f(x) \neq h(x)\}) \leq \frac{\delta}{1-3\delta}.$$

5.2 A description of the results from papers [E] **and** [F]

In paper [E] I prove a theorem similar to Theorem 5.1, but without the assumption (5.2), however, with the additional assumption that G and H are abelian ([E, Theorem 2.1]). Below I have rewritten a somehow more general version of this theorem: in the assumption (5.1) we demand that there are "few" pairs (x, y) such that not only the value $f(x + y)$ differs from the sum $f(x) + f(y)$ but is far enough from it.

Theorem 5.2 ([E] Theorem 2.2)**.** *Let G be a finite abelian group, Let H be an abelian group with translation-invariant metric $d\colon H \times H \to [0, \infty)$, $f\colon G \to H$, $\delta \in (0, 1 - \frac{\sqrt{3}}{2})$. Suppose that*

$$\mu^2(\{(x, y) :\ d(f(x) + f(y), f(x + y)) > \varepsilon\}) \leq \delta.$$

Then there is a 20ε-approximate homomorphism $h\colon G \to H$, i.e.,

$$d(h(a + b), h(a) + h(b)) \leq 20\varepsilon, \quad a, b \in G$$

such that

$$\mu(\{x :\ d(f(x), h(x)) > 7\varepsilon\}) \leq \frac{\delta}{1-2\delta}.$$

The aim of paper [F] is to remove the assumptions of the finiteness of G from theorems of the type presented above. According to M. M. Day [21] the existence of right invariant finitely additive probability measure defined on the set $\mathcal{P}(G)$ of all subsets of a group G is equivalent to the existence of right invariant mean on G. So we assume that G is amenable (with right invariant mean M) and that implies that there is a finitely aditive right invariant probability measure μ, and we have

$$\mu(A) = M(\chi_A), \quad A \subset G.$$

We can define in a natural way, that is by the formula[25]

$$\mu^2(Z) = M_y(\mu(Z^y)), \quad Z \subset G \times G,$$

[24] i.e., $\mu(A) = \frac{\operatorname{card} A}{\operatorname{card} G}$.

[25] In this formula lower index y denotes that $M_y(\mu(Z^y))$ is a value of invariant mean M taken on a function $y \mapsto \mu(Z^y)$ (with variable y), and $Z^y := \{x :\ (x, y) \in Z\}$.

finitely additive probability measure $\mu^2 \colon \mathcal{P}(G \times G) \to [0,1]$. It turns out that

$$\mu^2(A \times B) = \mu(A)\mu(B), \quad A, B \subset G.$$

So, we can generalize Theorem 5.2 to the case of amenable groups G, that is to the following theorem:

Theorem 5.3 ([F], Theorem 1.3)**.** *Let G be a group with a right-invariant, finitely additive probability measure $\mu : \mathcal{P}(G) \to [0,1]$. Suppose that H is a group with an invariant metric $d : H \times H \to [0, +\infty)$, $\varepsilon \geq 0$, $0 \leq \delta < \frac{1}{12}$. If a function $f : G \to H$ satisfies*

$$\mu^2\big(\{(x,y) \ : \ d\big(f(x+y), f(x) + f(y)\big) > \varepsilon\}\big) \leq \delta,$$

then for every $\zeta > 0$ there exists a map $F : G \to H$ such that

$$d\big(F(x+y), F(x) + F(y)\big) \leq 24\varepsilon, \ \ x, y \in G \tag{5.3}$$

and

$$\mu\big(\{x \ : \ d(f(x), F(x)) > \varepsilon\}\big) \leq 4\delta + \zeta.$$

The proof of theorem was divided into a few steps[26]. We defined the sets

$$Z = \{(x,y) \ : \ d(f(xy), f(x)f(y)) > \varepsilon\}$$

and (for a suitable $\eta > 0$)

$$\begin{aligned} U &= \{y \in G \ : \ \mu(\{x \in G \ : \ d(f(xy), f(x)f(y)) > \varepsilon\}) > \eta\} \\ &= \{y \in G \ : \ \mu(Z^y) > \eta\}. \end{aligned}$$

By estimating the measures of sets we deduced that for every $x \in G$ the set

$$A_x := G \setminus [U \cup (Ux^{-1})]$$

is nonempty and for every $x \in G \setminus U$ the set

$$B_x := G \setminus [U \cup (Ux^{-1}) \cup Z^x]$$

is nonempty.

That allowed us to choose y_x (for every $x \in G$) such that

$$y_x \in \begin{cases} A_x, & \text{if } x \in U, \\ B_x, & \text{if } x \in G \setminus U. \end{cases}$$

This enabled us to define $F : G \to H$ by the formula

$$F(x) = [f(y_x)]^{-1} f(y_x x), \quad x \in G.$$

[26]Full proof is quite long and laborous, it can be found on pages 516–519 of paper [F].

We showed that such F has all the desired properties.

Since every abelian group is an amenable group we can apply Theorem 5.3 to such groups, however, it turns out that with a separate proof (analogous to the proof of Theorem 5.2) we can get different thesis for abelian groups. Therefore we formulated and proved separately in [F] the following theorem:

Theorem 5.4 ([F], Theorem 3.1)**.** *Let G, H be commutative groups $\varepsilon \geq 0$, $0 \leq \delta \leq 1 - \frac{\sqrt{3}}{2}$. Assume that $\mu : \mathcal{P}(G) \to [0, 1]$ is an invariant, finitely additive probability measure on G and $d : H \times H \to [0, +\infty)$ is an invariant metric on a group H. If a function $f : G \to H$ satisfies*

$$\mu^2(\{(x, y) \ : \ d(f(x + y), f(x) + f(y)) > \varepsilon\}) \leq \delta,$$

then there is a map $F : G \to H$ such that

$$d(F(x + y), F(x) + F(y)) \leq 20\varepsilon, \quad x, y \in G \tag{5.4}$$

and

$$\mu(\{x \ : \ d(f(x), F(x)) > 7\varepsilon\}) \leq \frac{\delta}{1 - 2\delta}.$$

Both in Theorem 5.3 and in Theorem 5.4 we get the existence of a function $F\colon G \to H$ satisfying the additivity condition approximately (cf. (5.3), (5.4)).

If we add an assumption which will guarantee the stability of the Cauchy equation we will be able to deduce that there exists an additive function such that for majority of points (in a sense of measure μ) values of this additive function and of function f differ only slightly.

For the clarity let us use the Hyers Theorem to get the following result:

Corollary 5.1 ([F], Corollary 3.2)**.** *Assume that $\varepsilon \geq 0$, $0 \leq \delta \leq 1 - \frac{\sqrt{3}}{2}$, $\mu : \mathcal{P}(G) \to [0, 1]$ is an invariant, finitely additive probability measure on a commutative group G, let a Y be a Banach space. If a function $f : G \to Y$ satisfies*

$$\mu^2(\{(x, y) \in G^2 \ : \ \|f(x + y) - (f(x) + f(y))\| > \varepsilon\}) \leq \delta,$$

then there exists a function $h : G \to Y$ such that

$$h(x + y) = h(x) + h(y), \quad x, y \in G,$$

and

$$\mu(\{x \in G \ : \ \|f(x) - h(x)\| > 27\varepsilon\}) \leq \frac{\delta}{1 - 2\delta}.$$

In the case of amenable groups we can use, for example, [26, Theorem 3] and [106], to get the following corollary:

Corollary 5.2 ([F], Corollary 3.3)**.** *Let G be a group with a right-invariant, finitely additive probability measure $\mu : \mathcal{P}(G) \to [0,1]$, Y a Banach space, $\varepsilon \geq 0$, $0 \leq \delta < \frac{1}{12}$. If a function $f : G \to Y$ satisfies*

$$\mu^2\big(\{(x,y) \;:\; \|f(x+y) - f(x) - f(y)\| > \varepsilon\}\big) \leq \delta,$$

then for every $\zeta > 0$ there exists a map $F : G \to Y$ such that

$$F(x+y) = F(x) + F(y), \quad x, y \in G,$$

and

$$\mu\big(\{x \;:\; \|f(x) - F(x)\| > 25\varepsilon\}\big) \leq 4\delta + \zeta.$$

5. Presentation of other research achievements

The list of publications not included in the habilitation thesis and written after Ph.D. :

[I] Barbara Przebieracz, *On the stability of the translation equation*, Publicationes Mathematicae Debrecen, 75 (2009), 285–298.

[II] Roman Badora, Barbara Przebieracz, Peter Volkmann, *Stability of the Pexider functional equation*, Annales Math. Silesianae 24 (2010), 7–13.

[III] Barbara Przebieracz, *Superstability of some functional equation*, Series Mathematicae Catoviciensis et Debreceniensis, No. 31, (2010), 4 pp., http://www.math.us.edu.pl/smdk.

[IV] Barbara Przebieracz, *Stability of the Baron-Volkmann functional equations*, Math. Inequalities and Applications 14(1) (2011), 193–201.

[V] Barbara Przebieracz, *The stability of the functional equation* $\min\{f(x+y), f(x-y)\} = |f(x)-f(y)|$, Journal of Inequalities and Applications 2011:22 (2011).

[VI] Barbara Przebieracz, *On some Pexider-type functional equations connected with the absolute value of additive functions. Part I*, Bull. Aust. Math. Soc. 85, No. 2 (2012), 191–201.

[VII] Barbara Przebieracz, *On some Pexider-type functional equations connected with the absolute value of additive functions. Part II*, Bull. Aust. Math. Soc. 85, No. 2 (2012), 202–216.

[VIII] Roman Badora, Barbara Przebieracz, Peter Volkmann, *On Tabor groupoids and stability of some functional equations*, Aequationes Math. 87 (2014), 165–171.

[IX] Roman Badora, Barbara Przebieracz, Peter Volkmann, *Stability of generalized Cauchy equations.* Aequationes Math. 89 (1) (2015), 49–56.

[X] Zenon Moszner, Barbara Przebieracz, *Is the dynamical system stable?*. Aequationes Math. 89 (2) (2015), 279–296.

[XI] Barbara Przebieracz, *A proof of the Mazur-Orlicz theorem via the Markov-Kakutani common fixed point theorem, and vice versa*, Fixed Point Theory Appl. 2015, 2015:10, 9 pp.

[XII] Roman Badora, Barbara Przebieracz, Peter Volkmann, *More on Hyers' Theorem*, Journal of Math. Analysis and Applications, 447 (2) (2017), 1116–1125.

[XIII] Barbara Przebieracz, *Recent Developments in the translation equation and its stability*, In: J. Brzdęk, et al. (eds) Developments in Functional Equations and Related Topics. Springer International Publishing, Cham (2017), 215–229.

6. The equations characterizing the absolute value of an additive function

Introduction

In this part I am going to describe the results concerning the solutions and the stability of the following functional equations:

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y), \tag{6.1}$$

$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)|, \tag{6.2}$$

$$\max\{f(x+y),f(x-y)\}=f(x)f(y), \tag{6.3}$$

$$\sup\{f(x+\lambda y):\ \lambda\in T\}=f(x)+f(y), \tag{6.4}$$

$$\inf\{f(x+\lambda y):\ \lambda\in T\}=|f(x)-f(y)|, \tag{6.5}$$

$$\sup\{f(x+\lambda y):\lambda\in T\}=f(x)f(y), \tag{6.6}$$

$$\sup\{f(x+l(y)):\ l\in L\}=f(x)f(y), \tag{6.7}$$

$$\min\{f(x+y),f(x-y)\}=f(x)f(y), \tag{6.8}$$

$$\max\{f(x+y),f(x-y)\}=g(x)h(y), \tag{6.9}$$

$$\max\{f(x+y),f(x-y)\}=f(x)g(y)+h(y), \tag{6.10}$$

$$\max\{f(x+y),f(x-y)\}=f(y)g(x)+h(x). \tag{6.11}$$

In equations (6.1), (6.2), (6.3), (6.8), (6.9), (6.10), (6.11) the real functions f,g,h are defined on an abelian group G; in equations (6.4), (6.5) and (6.6) a real function f is defined on a vector space V (over the field $\mathbb{C}$), and T denotes the unit circle in $\mathbb{C}$; in equation (6.7) a real function f is define on an abelian group G, $L\subset G^G$, $\mathrm{id},-\mathrm{id}\in L$.

My contribution to establishing the form of the solutions or the stability of these equations is included in the papers [III]–[VIII].

• **About the equation** (6.1)

As it was proved by A. Simon and P. Volkmann in [103], the equation (6.1) characterizes the absolute value of additive functions (other proofs were given by T. Kochanek and [65] and W. Fechner [25]). This result was also achieved by P. Volkmann in [118] without the assumption of the commutativity of G, but with a weaker assumption: $f(xyz)=f(yxz)$ for $x,y,z\in G$. This assumption was removed by I. Toborg in [115].

The proof of the stability of this equation can be found in [52]. The proof of the stability of a more general equation

$$\max\{f((x\circ y)\circ y);f(x)\}=f(x\circ y)+f(y), \tag{6.12}$$

where a real function is defined on a groupoid G with binary operation $\circ$ such that

$$(x\circ y)\circ(x\circ y)=(x\circ x)\circ(y\circ y),\quad x,y\in G,$$

and with left neutral element can be found in [38]; whereas the paper [VIII] is devoted to the stability of equation (6.12) with even more weaker assumptions concerning the domain of f (G is a groupoid with a binary operation $\circ$ such that for every $x,y\in G$ there exists a $k\in\mathbb{N}$ such that

$$(x\circ y)^{2^k}=x^{2^k}\circ y^{2^k},\quad ((x\circ y)\circ y)^{2^k}=(x^{2^k}\circ y^{2^k})\circ y^{2^k},$$

the powers x^{2^k} are defined recursively by: $x^{2^0} := x$, $x^{2^{k+1}} := x^{2^k} \circ x^{2^k}$).

Furthermore, the stability of the equation (6.1) was proved in [112] as a corollary from the stability of a more general functional equation

$$f(x \circ y) \star f(x \diamond y) = f(x \circ y).$$

• **About the equation** (6.2)

The form of the solutions to the equation (6.2) under the assumption that $G = \mathbb{R}$, and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function was given in the following theorem:

Theorem 6.1 ([52]). *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (6.2). *Then either there exists a constant $c \geq 0$ such that $f(x) = c|x|$, $x \in \mathbb{R}$, or f is periodic with a period $2p$ and $f(x) = c|x|$ for $x \in [-p, p]$, with some constant $c > 0$.*

Actually, it is enough to assume the continuity in at least one point, since it implies the continuity of f on the whole $\mathbb{R}$ ([52]). Moreover, some measurability assumptions concerning f imply its continuity, which was investigated in [7] and [67].

Furthermore, T. Kochanek noticed that every function $f = g \circ a$ defined on an abelian group G, where $g\colon \mathbb{R} \to \mathbb{R}$ is a solution to the equation (6.2) described in Theorem 6.1, and $a\colon G \to \mathbb{R}$ is an additive function, is a solution to the equation (6.2).

In paper [V] there is a proof of stability of equation (6.2) in the class of real continuous functions defined on $\mathbb{R}$, i.e., a proof of the following theorem:

Theorem 6.2. *Let $\delta \geq 0$ and $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$|\min\{f(x+y), f(x-y)\} - |f(x) - f(y)|| \leq \delta, \quad x, y \in \mathbb{R},$$

then either f is bounded (and in this case it is close to the solution $F \equiv 0$ to the equation (6.2)) *or there exists a constant $c > 0$ such that*

$$|f(x) - c|x|| \leq 21\delta, \quad x \in \mathbb{R},$$

i.e., f is close to the solution $F(x) = c|x|$ to the equation (6.2).

• **About the equations** (6.3) **and** (6.7)

The form of the solutions to the equation (6.3) under the additional assumption that the abelian group G is divided by 6 were presented in [103] (these are the functions of the form (i) or (ii) from the theorem below), and without this additional assumption in [VI]:

Theorem 6.3. *Let G be an abelian group and $f\colon G \to \mathbb{R}$. Then f satisfies the functional equation* (6.3) *if and only if*

(i) $f \equiv 0$

or

(ii) $f = exp \circ |a|$, *for an additive function* $a\colon G \to \mathbb{R}$

or

(iii) *there exists a subgroup* G_0 *of group* G *such that*

$$x, y \in G \setminus G_0 \Rightarrow (x + y \in G_0 \vee x - y \in G_0),$$

and

$$f(x) = \begin{cases} 1, & x \in G_0; \\ -1, & x \notin G_0. \end{cases}$$

Superstability of the equation (6.3) follows from the superstability of a more general form of this equation, i.e., the equation (6.7) which was proved in [III]. Namely, using the ideas from [5] I proved the following theorem:

Theorem 6.4. *Let* $f\colon G \to \mathbb{R}$ *satisfy the inequality*

$$|\sup\{f(x + l(y));\, l \in L\} - f(x)f(y)| \leq \varepsilon, \qquad x, y \in G,$$

then either f *is bounded or it is a solution to* (6.7).

• **About the equations** (6.4) **and** (6.5)

The equations (6.4) and (6.5) are analogous to (6.1) and (6.2), respectively, for functions defined on some vector spaces over $\mathbb{C}$. It turned out (see [8, Theorem 1]) that the equations (6.4) and (6.5) are equivalent, moreover, each of them characterizes the absolute value of linear functional.

In [IV] I investigated the stability of (6.4) and (6.5):

Theorem 6.5. *Let* $\delta \geq 0$, $g\colon V \to \mathbb{R}$ *satisfy*

$$|\sup_{\lambda \in T} g(x + \lambda y) - g(x) - g(y)| \leq \delta, \quad x, y \in V.$$

Then there is a solution $f\colon V \to \mathbb{C}$ *to the equation* (6.4) *such that*

$$|f(x) - g(x)| \leq 17\delta, \quad x \in V.$$

Theorem 6.6. *Let* $\delta \geq 0$, $g\colon V \to \mathbb{R}$ *satisfy*

$$|\inf_{\lambda \in T} g(x + \lambda y) - |g(x) - g(y)|| \leq \delta, \quad x, y \in V.$$

Then there exists a solution $f\colon V \to \mathbb{C}$ *to the equation* (6.5) *such that*

$$|f(x) - g(x)| \leq 49\delta, \quad x \in V.$$

• **About the equation** (6.6)

The form of the solutions to the equation (6.6) was described in [III]:

Theorem 6.7. *If* $f\colon V \to \mathbb{R}$ *satisfies the functional equation* (6.6) *then either* $f \equiv 0$ *or there exists a linear functional* $\phi\colon V \to \mathbb{C}$ *such that* $f(x) = \exp|\phi(x)|$, $x \in V$.

Superstability of this equation follows from Theorem 6.4 ([III, Theorem 1.1]).

• **About the equation** (6.8)

The following result concerning the form of the solutions to the equation (6.8) comes from [VI].

Theorem 6.8. *Let* G *be an abelian group and* $f\colon G \to \mathbb{R}$. *Then* f *is a solution to the functional equation* (6.8) *if and only if it has one of the following form:*

1. $f \equiv 0$,

or

2. $f(x) = \exp(-|a(x)|)$, $x \in G$, *for an additive* $a\colon G \to \mathbb{R}$,

or

3. there is a subgroup G_0 *of* G *with the property:*

$$x, y \in G \setminus G_0 \;\Rightarrow\; (x + y \in G_0 \wedge x - y \in G_0), \tag{6.13}$$

and

$$f(x) = \begin{cases} 1, & x \in G_0, \\ -1, & x \notin G_0, \end{cases}$$

or

4. there is a subgroup G_0 *of* G *with the property:*

$$x, y \in G \setminus G_0 \;\Rightarrow\; (x + y \notin G_0 \vee x - y \notin G_0), \tag{6.14}$$

and

$$f(x) = \begin{cases} 1, & x \in G_0, \\ 0, & x \notin G_0. \end{cases}$$

• **About the equation** (6.9)

As a corollary from [VII, Theorem 3.1] we get in [VI] the form of the solutions on the real line, to the functional equation (6.9):

Theorem 6.9. *Let* $f, g, h\colon \mathbb{R} \to \mathbb{R}$ *and* f *be continuous. Then*

$$\max\{f(x+y), f(x-y)\} = g(x)h(y) \quad \textit{for } x, y \in \mathbb{R}$$

if and only if one of the following holds:

1. $f \equiv 0$, $g \equiv 0$, h *– an arbitrary;*

2. $f \equiv 0, \quad h \equiv 0, \quad g -$ *anarbitrary;*
3. $f(x) = bce^{a|x-x_0|}, \quad g(x) = be^{a|x-x_0|}, \quad h(x) = ce^{a|x|}$, $x \in \mathbb{R}$, *where* $a, b, c, x_0 \in \mathbb{R}$, $abc > 0$*;*
4. $f(x) = bce^{ax}, \quad g(x) = be^{ax}, \quad h(x) = ce^{\mathrm{sgn}(bc)|ax|}$, $x \in \mathbb{R}$, *where* $a, b, c \in \mathbb{R}$.

• **About the equations** (6.10) **and** (6.11)

The equation

$$f(x+y) = f(x)g(y) + h(y).$$

investigated in the monograph [2] was an ispiration for the papers [VI] and [VII]. It turns out the the solutions to the above equations are either "trivial" with f being constant, or connected with the solutions to the equation

$$a(x+y) = a(x) + a(y),$$

or connected to the solutions to the equation

$$e(x+y) = e(x)e(y).$$

Similarly I consider the common "pexiderization" of equations (6.1) and (6.3), however, since the roles of x and y differ, I have got two functional equations: (6.10) and (6.11). The main result of the paper [VI] describes the form of the solutions to the equation (6.11). Suprisingly, it turns out that aside from the analogous results to these from [2], i.e., trivial solutions, with constant f, solutions connected with the equation (6.1) and solutions connected with the equation (6.3), we get one more type of solutions, namely, connected with the equation (6.8) (and that is why I have investigated also such an equation).

Theorem 6.10. *Let* $f, g, h \colon G \to \mathbb{R}$, G *be an abelian group. Then* f, g, h *satisfy the functional equation* (6.11) *if and only if one of the following possibilities holds:*
1. $f(x) = b, \quad g -$ *an arbitrary,* $\quad h(x) = b(1 - g(x))$, $x \in G, \quad$ *where* $b \in \mathbb{R}$*;*
2. $f(x) = c\phi(x) + b, \quad g(x) = \phi(x), \quad h(x) = b(1 - \phi(x))$, $x \in G, \quad$ *where* $c, b \in \mathbb{R}$, $c > 0$, *and* $\phi \colon G \to \mathbb{R}$ *satisfies* (6.3)*;*
3. $f(x) = c\phi(x) + b, \quad g(x) = \phi(x), \quad h(x) = b(1 - \phi(x))$, $x \in G, \quad$ *where* $c, b \in \mathbb{R}$, $c < 0$, *and* $\phi \colon G \to \mathbb{R}$ *satisfies* (6.8)*;*
4. $f(x) = \phi(x) + b, \quad g(x) = 1, \quad h(x) = \phi(x)$, $x \in G, \quad$ *where* $b \in \mathbb{R}$, *and* $\phi \colon G \to \mathbb{R}$ *satisfies* (6.1).

The general form of (f, g, h) satisfying (6.10) was presented in [VII] under the additional assumption that f, g, h are defined on $\mathbb{R}$ and that f is continuous:

Theorem 6.11. *Let $f, g, h\colon \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (6.10). *If f is continuous then we have one of the following possiblities:*
1. $f(x) = b$, $g -$ *an arbitrary function,* $h(x) = b(1 - g(x))$, $x \in \mathbb{R}$, *where* $b \in \mathbb{R}$;
2. $f(x) = ce^{a|x-x_0|} + b$, $g(x) = e^{a|x|}$, $h(x) = b(1 - e^{a|x|})$, $x \in \mathbb{R}$, *where* $x_0, b \in \mathbb{R}$, $ac > 0$;
3. $f(x) = a|x - x_0| + b$, $g(x) = 1$, $h(x) = a|x|$, $x \in \mathbb{R}$, *where* $b, x_0 \in \mathbb{R}$, $a > 0$;
4. $f(x) = ce^{ax} + b$, $g(x) = e^{\mathrm{sgn}(c)|ax|}$, $h(x) = b(1 - e^{\mathrm{sgn}(c)|ax|})$, $x \in \mathbb{R}$, *where* $a, b, c \in \mathbb{R}$;
5. $f(x) = ax + b$, $g(x) = 1$, $h(x) = |ax|$, $x \in \mathbb{R}$, *where* $a, b \in \mathbb{R}$.

And conversely, if f, g, h are of the forms described in one of the points 1-5 then (f, g, h) satisfies (6.10).

7. New proofs of Mazur–Orlicz Theorem and Markov–Kakutani Theorem

7.1 Introduction

Many well known and important theorems have been proved in many different ways. The same applies to theorems of Markov-Kakutani and Mazur-Orlicz. In paper [XI] I show the direct connection between these theorems by proving one with the use of the other.

7.2 Mazur-Orlicz Theorem and its proofs

Let me remind the Mazur-Orlicz Theroem:

Theorem 7.1. *(*[80]*) Let X be a linear space, T a nonempty set, $x\colon T \to X$, $\beta\colon T \to \mathbb{R}$, and $p\colon X \to \mathbb{R}$ be a sublinear functional. Then the following conditions are equivalent:*

(i) *there exists a linear functional $a\colon X \to \mathbb{R}$ such that*

$$a(y) \leq pyx), \quad y \in X,$$

$$\beta(t) \leq a(x(t)), \quad t \in T;$$

(ii) *for every $n \in \mathbb{N}$, $t_1, \ldots, t_n \in T$ i $\lambda_1, \ldots, \lambda_n \in (0, \infty)$,*

$$\sum_{i=1}^{n} \lambda_i \beta(t_i) \leq p\left(\sum_{i=1}^{n} \lambda_i x(t_i)\right).$$

Let me remind also the counterpart of this theorem for abelian groups:

Theorem 7.2. *Let G be an abelian group, T a nonempty set, $x\colon T \to G$, $\beta\colon T \to \mathbb{R}$, and $p\colon G \to \mathbb{R}$ subadditive. Then the following conditions are equivalent:*

(i) *there exists an additive function $a\colon G \to \mathbb{R}$ such that*

$$a(y) \leq p(y), \quad y \in G,$$

$$\beta(t) \le a(x(t)), \quad t \in T;$$

(ii) *for every* $n \in \mathbb{N}$ *and* $t_1, \dots, t_n \in T$,

$$\sum_{i=1}^{n} \beta(t_i) \le p\left(\sum_{i=1}^{n} x(t_i)\right).$$

In both versions the implication (i)⇒ (ii) is obvious.

Except the long and rather difficult proof from [80], there are many different proofs of Mazur-Orlicz Theorem and its generalizations. It can be found for example in [15], [19], [63], [74], [92], [95] (probably the most elementary and elegant), [102] and [104].

My proof presented in [XI] relies on Markov-Kakutani Theorem. It lets us prove both Theorem 7.1 and 7.2 in an analogous way. It does not need any other sophisticated theorems to be used[27].

7.3 Markov-Kakutani Theorem and its proofs

Let me remind (once more) Markov-Kakutani Theorem.

Theorem 7.3. *(A. Markov* [79]*, S. Kakutani* [58]*) Let X be a (locally convex*[28]*) linear-topological space, $\mathcal{C}$ a nonempty convex compact subset of X, $\mathcal{F}$ a commuting family of continuous affine selfmappings of $\mathcal{C}$. Then there exists an $x \in \mathcal{C}$ such that $f(x) = x$ for every $f \in \mathcal{F}$.*

Markov-Kakutani Theorem can be proved with the use of different corollaries from Hahn-Banach Theorem. In [119], Markov-Kakutani Theorem was proved by separation theorem (nonvoid compact convex disjoint sets can be strongly separeted in locally convex spaces). Also from separation theorem (point can be separated in locally convex spaces) and from already mentioned [40, Theorem 3.2.2], Markov-Kakutani Theorem is derived in [40]. One more proof of Markov-Kakutani Theorem based on separation theorem can be found in [98].

Let me mention that the proof of Markov-Kakutani Theorem from [58] can be found in monograph [99], whereas the most elegant and elementary one in my view belongs to J. Jachymski [50].

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[27] It may seem that also in [40] Mazur-Orlicz Theorem was proved by the use of Markov-Kakutani Theorem. To emphasize the difference between the proof from [40] and the proof from [XI] let me describe shortly the proof from [40]. First, in [40, Lemma 4.5.1] (repeating the result from [58]) some corollary from Markov-Kakutani Theorem was used to prove Hahn-Banach Theorem. However, to prove Mazur-Orlicz Theorem, authors of [40] use also some lemma about supporting sublinear functionals by linear functionals and yet one more important result from the theory of infinite systems of inequalities [40, Theorem 3.2.2]

[28] This assumption is not necessary, however, many known proofs of this theorem do not go beyond such spaces.

The proof of Markov-Kakutani Theorem presented in [XI] is derived directly from Mazur-Orlicz Theorem, and, similarly as in [40, 79, 98, 119] requires locally compact linear-topological spaces.

7.4 The sketch of the proof of implication (ii) $\Rightarrow$ (i) **of Mazur-Orlicz Theorem (presented in** [XI]**)**

• In the case of Theorem 7.2 we consider $\mathbb{R}^G$ equipped with the product topology and maps $F_y\colon \mathbb{R}^G \to \mathbb{R}^G$, $y \in G$, defined by

$$F_y f(z) = f(z+y) - f(y), \quad z \in G, f \in \mathbb{R}^G.$$

The maps F_y are continuous and affine and

$$F_y \circ F_z = F_{y+z}, \quad y, z \in G, \tag{7.1}$$

so the family $\{F_y; y \in G\}$ is commuting. Let us choose $t_0 \in T$ and put $s := p(x_{t_0}) - \beta_{t_0}$ and

$$\begin{aligned}\mathcal{C} = \{f \in \mathbb{R}^G : &- p(-y) \leq f(y) \leq p(y) + s,\, y \in G,\\ &- p(-y) \leq F_z f(y) \leq p(y),\, y, z \in G,\\ &\beta(t) \leq f(x(t)), t \in T,\\ &\beta(t) \leq F_y f(x(t)), t \in T, y \in G\}.\end{aligned}$$

From Markov-Kakutani Theorem we infer that there exists an $a \in \mathcal{C}$ being a fixed point of every F_y, for $y \in G$. It means that

$$a(y+z) = a(y) + a(z), \quad y, z \in G,$$

$$\beta(t) \leq a(x(t)), \quad t \in T$$

and

$$a(y) \leq p(y) + s, \quad y \in G.$$

From the last inequality we can deduce that

$$a(y) \leq p(y), \quad y \in G.$$

• In the case of Theorem 7.1 similarly like in the proof of Theorem 7.2 (described above) we get the existence of an appropriate additive function a and one can show easily that this a is also linear.

7.5 The sketch of the first step of the proof of Markov-Kakutani Theorem (presented in [XI]**)**

Often the proof of Markov-Kakutani Theorem is divided into two steps. First, the following theorem is shown[29]:

Theorem 7.4. *Let X be a locally convex linear-topological space, $\mathcal{C} \subset X$ be a nonvoid convex compact set and $F\colon \mathcal{C} \to \mathcal{C}$ continuous and affine function. Then F has a fixed point.*

We can use either Theorem 7.1 or Theorem 7.2. Let us put

$$B = \{F(x) - x : x \in \mathcal{C}\}.$$

Suppose that $0 \notin B$. The set B is nonempty convex and compact. Let $U \subset X \setminus B$ be a convex and balanced neighbourhood of zero. Of course, U is absorbing. Let $p\colon X \to \mathbb{R}$ be a Minkowski functional of the set U, i.e.,

$$p(x) = \inf\{r > 0;\, x \in rU\}, \quad x \in X.$$

The functional p is sublinear and $\{x \in X;\, p(x) < 1\} \subset U$. From the compactness of $\mathcal{C}$ we infer that there exists an $N \in \mathbb{N}$ such that $\mathcal{C} \subset NU$, and hence

$$p(x) \le N, \quad x \in \mathcal{C}. \tag{7.2}$$

Let $T = \mathcal{C}$ and $\beta(t) = 1$, $x(t) = F(t) - t$ for $t \in T$. For an arbitrary $n \in \mathbb{N}$, $t_1, \ldots, t_n \in T$ and $\lambda_1, \ldots, \lambda_n \in (0, \infty)$, we have

$$\sum_{i=1}^{n} \lambda_i \beta(t_i) \le p\left(\sum_{i=1}^{n} \lambda_i x(t_i)\right).$$

So the condition (ii) of Mazur-Orlicz Theorem is satisfied. Therefore there exists a linear functional $a\colon X \to \mathbb{R}$ such that

$$a(x) \le p(x), \quad x \in X$$

and

$$1 = \beta(t) \le a(x(t)) = a(F(t) - t), \quad t \in \mathcal{C}.$$

For $x \in \mathcal{C}$ and $n \in \mathbb{N}$ one can check that

$$a(F^n(x)) \ge a(x) + n, \quad x \in \mathcal{C}, n \in \mathbb{N},$$

which taking into account (7.2), implies that

$$N \ge p(F^n(x)) \ge a(F^n(x)) \ge a(x) + n, \quad x \in \mathcal{C}, n \in \mathbb{N}.$$

We get a contradiction. Hence $0 \in B$, which means that there exists a fixed point of F.

[29]The second step (i.e., proof of the Markov-Kakutani Theorem using Theorem 7.4) which I presented in [XI], is standard, the same can be found in [40], [119], [98], [50], hence, I will not repeat it here.

8. Results concerning stability of the translation equation and dynamical systems not included in the habilitation thesis

8.1 Introduction

Connections between various approaches to the problem of stability of the translation equation (see [83], [86], [88], [89], [90]) and selected paradoxes related to this topic ([78], [90]) are among many research interests of Z. Moszner. In a shared article [X] we investigated whether the systems of equations which define dynamical system or the translation equation in some classes of functions (such classes in which the solution of the translation equation is a dynamical system) are stable (and we took into consideration different types of stability, most of them I will describe in what follows). Further results concerning this topic can be found in [89] and [90].

8.2 Dynamical systems

Let $I \subset \mathbb{R}$ be a nondegenerate real interval and $F\colon \mathbb{R} \times I \to I$. By F^0 we denote the function $F(0, \cdot)$.

Defintion 8.1. The continuous function $F\colon \mathbb{R} \times I \to I$ is called *a dynamical system*, if it is a solution to the translation equation:

$$F(t, F(s, x)) = F(s + t, x), \quad s, t \in \mathbb{R},\, x \in I; \tag{8.1}$$

and

$$F(0, x) = x, \quad x \in I. \tag{8.2}$$

It turns out that the condition (8.2) in the above definition can be replaced by any of the following:

(1) function F^0 is differentiable and

$$(F^0)'(x) = 1, \quad x \in I. \tag{8.3}$$

(2) F^0 is strictly increasing.
(3) F is not constant and F^0 is differentiable.
(4) F is a surjection.

In a paper [X] we write, for the sake of simplicity, about the dynamical systems according to one out of the five (equivalent) definitions.

8.3 Different kinds of stability

To simplify the following definitions let us introduce some labels:

$$|H(s, H(t, x)) - H(t + s, x)| \le \delta, \quad s, t \in \mathbb{R},\, x \in I; \tag{8.4}$$

$$|H(0,x)-x| \le \delta, \quad x \in I; \tag{8.5}$$

$$|(H^0)'(x)-1| \le \delta, \quad x \in I. \tag{8.6}$$

Furthermore put

$$\mathcal{H} := \{H \colon \mathbb{R}\times I \to I : \ H \text{ continuous}\};$$

$\mathcal{F}_j := \{F \colon \mathbb{R}\times I \to I : \ F$ is a continuous solution of the system: (8.1) & (8.j)$\}$ for $j = 2, 3$;
$\mathcal{H}_k := \{H \colon \mathbb{R}\times I \to I : \ H$ is a continuous solution of the system: (8.4) & (8.k)$\}$ for $k = 5, 6$.
We say that the system of functional equations:(8.1) & (8.j), for $j = 2, 3$, is:

- *Hyers-Ulam stable*, if

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{H\in\mathcal{H}} \ \left[\text{if } H \in \mathcal{H}_{j+3}, \text{ then } \exists_{F\in\mathcal{F}_j}\, |H(s,x)-F(s,x)| \le \varepsilon \text{ for } s \in \mathbb{R},\, x \in I\right];$$

- *b-stable*, if

$$\forall_{\delta>0}\, \forall_{H\in\mathcal{H}} \ \left[\text{if } H \in \mathcal{H}_{j+3}, \text{ then } \exists_{\varepsilon>0}\, \exists_{F\in\mathcal{F}_j}\, |H(s,x)-F(s,x)| \le \varepsilon \text{ for } s \in \mathbb{R},\, x \in I\right];$$

- *uniformly b-stable*, if

$$\forall_{\delta>0}\, \exists_{\varepsilon>0}\, \forall_{H\in\mathcal{H}} \ \left[\text{if } H \in \mathcal{H}_{j+3}, \text{ then } \exists_{F\in\mathcal{F}_j}\, |H(s,x)-F(s,x)| \le \varepsilon \text{ for } s \in \mathbb{R},\, x \in I\right];$$

- *inversely stable*, if

$$\forall_{\delta>0}\, \exists_{\varepsilon>0}\, \forall_{H\in\mathcal{H}} \ \left[\text{if } \exists_{F\in\mathcal{F}_j}\, |H(s,x)-F(s,x)| \le \varepsilon \text{ for } s \in \mathbb{R},\, x \in I, \text{ then } H \in \mathcal{H}_{j+3}\right];$$

- *inversely b-stable*, if

$$\forall_{H\in\mathcal{H}} \ \left[\text{if } \exists_{\varepsilon>0}\, \exists_{F\in\mathcal{F}_j}\, |H(s,x)-F(s,x)| \le \varepsilon \text{ for } s \in \mathbb{R},\, x \in I, \text{ then } \exists_{\delta>0}\, H \in \mathcal{H}_{j+3}\right];$$

- *inversely uniformly b-stable*, if

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{H\in\mathcal{H}} \ \left[\text{if } \exists_{F\in\mathcal{F}_j}\, |H(s,x)-F(s,x)| \le \varepsilon \text{ for } s \in \mathbb{R},\, x \in I, \text{ then } H \in \mathcal{H}_{j+3}\right].$$

Below I will formulate the definitions of different types of stability of the translation equation in the class of functions F such that F^0 is strictly increasing. If we change the expression "F^0 is strictly increasing" into either "F is not constant and $(F^0)'$ exists" or "F is a surjection" in the definitions below, we obtain definitions of stability of the translation equation in, respectively, the class of functions F such that F is not constant and $(F^0)'$ exists, or the class of surjective functions. Here also, to simplify some expressions we introduce some notation. Let

$$\mathcal{K} := \{F \colon \mathbb{R}\times I \to I : \ F \text{ continuous}, \ F^0 \ \text{ is strictly increasing}\},$$

$$\mathcal{K}_1 := \{F \in \mathcal{K} : \ F \text{ is a continuous solution of } \ (8.1)\},$$

$$\mathcal{K}_4 := \{H \colon \mathbb{R}\times I \to I : \ H \text{ is a continuous solution of } \ (8.4)\}.$$

We say that the translation equation (8.1) is

- *stable in Hyers-Ulam sense in the class* $\mathcal{K}$, if

$$\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{H\in\mathcal{K}}\ \left[\text{if }\ H\in\mathcal{K}_4,\ \text{ then }\exists_{F\in\mathcal{K}_1}\,|H(s,x)-F(s,x)|\le\varepsilon\text{ for }s\in\mathbb{R},\,x\in I\right];$$

- *b-stable in the class* $\mathcal{K}$, if

$$\forall_{\delta>0}\,\forall_{H\in\mathcal{K}}\ \left[\text{if }H\in\mathcal{K}_4,\ \text{ then }\exists_{\varepsilon>0}\,\exists_{F\in\mathcal{K}_1}\ |H(s,x)-F(s,x)|\le\varepsilon\text{ for }s\in\mathbb{R},\,x\in I\right];$$

- *uniformly b-stable in the class* $\mathcal{K}$, if

$$\forall_{\delta>0}\,\exists_{\varepsilon>0}\,\forall_{H\in\mathcal{K}}\ \left[\text{if }H\in\mathcal{K}_4,\ \text{ then }\exists_{F\in\mathcal{K}_1}\ |H(s,x)-F(s,x)|\le\varepsilon\text{ for }s\in\mathbb{R},\,x\in I\right];$$

- *inversely stable in the class* $\mathcal{K}$, if

$$\forall_{\delta>0}\,\exists_{\varepsilon>0}\,\forall_{H\in\mathcal{K}}\ \left[\text{if }\ \exists_{F\in\mathcal{K}_1}\ |H(s,x)-F(s,x)|\le\varepsilon\text{ for }\ s\in\mathbb{R},\,x\in I,\ \text{ then }H\in\mathcal{K}_4\right];$$

- *inversely b-stable in the class* $\mathcal{K}$, if

$$\forall_{H\in\mathcal{K}}\ \left[\text{if }\exists_{\varepsilon>0}\,\exists_{F\in\mathcal{K}_1}\ |H(s,x)-F(s,x)|\le\varepsilon\text{ for }s\in\mathbb{R},\,x\in I,\ \text{ to }\exists_{\delta>0}\,H\in\mathcal{K}_4\right];$$

- *inversely uniformely b-stable in the class* $\mathcal{K}$, if

$$\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{H\in\mathcal{K}}\ \left[\text{if }\exists_{F\in\mathcal{K}_1}\ |H(s,x)-F(s,x)|\le\varepsilon\text{ for }s\in\mathbb{R},\,x\in I,\ \text{ then }H\in\mathcal{K}_4\right].$$

8.4 Summary of the results obtained in [X]

The table below summarises the results of investigations concerning different kinds of stability of dynamical systems (several others were discussed in the article). It turned out that the stability of the dynamical system depends on the boundness of interval I and on which of the equivalent five definitions of dynamical system we consider.

<table>
<tr><td></td><td>def.1
((8.1) &
$F(0,x)=x$)</td><td>def .2
((8.1) &
$(F^0)'(x)=1$)</td><td>def. 3
((8.1) &
$F(0,\cdot)$ strictly increasing)</td><td>def. 4
((8.1) &
$(F^0)'$ exists)</td><td>def. 5
((8.1) &
F sujection)</td></tr>
<tr><td>stability in a sense Hyers-Ulam</td><td>only for $I=\mathbb{R}$</td><td>for every I</td><td colspan="2">for no I</td><td>for every I</td></tr>
<tr><td>b-stability</td><td rowspan="2">only for I bounded or $I=\mathbb{R}$</td><td rowspan="2" colspan="3">only for I bounded</td><td rowspan="2">for every I</td></tr>
<tr><td>uniform b-stability</td></tr>
<tr><td>inverse stability</td><td colspan="5">for no I</td></tr>
<tr><td>inverse b-stability</td><td rowspan="2">only for I bounded</td><td rowspan="2">for no I</td><td rowspan="2" colspan="3">only for I bounded</td></tr>
<tr><td>inverse uniform b-stability</td></tr>
</table>

8.5 The stability of the translation equation in the class of continuous function $F\colon (0,\infty)\times I\to I$

At the end of this section let me discuss the results from [I]. I investigated there the stability of the translation equation in the class of continuous functions defined on $(0,\infty)\times I$ with values in I, where $I\subset\mathbb{R}$ is an interval. The full answer to the question analogous to that one posed by Ulam was not achieved. Therefore this article includes only some partial results. In [I, Theorem 3.1] I listed some additional conditions under which an approximate iteration semigroup is close to some iteration semigroup. And in [I, Theorem 3.2] I proved without those additional conditions that δ-iteration semigroup $H\colon (0,\infty)\times I\to I$ can be approximated (with ε accuracy) by an iteration semigroup not necessarily on the whole interval I, but at least on the set $\operatorname{cl} H((0,\infty)\times I)\setminus L$, where $|L|\le\eta$ (δ is suitably chosen to the given arbitrary ε and $\eta>0$).

9. Stability of Cauchy and Pexider equations

9.1 Results from [II]

In [II] we proved the stability of the Pexider functional equation

$$F(xy)=G(x)+H(y),\quad x,y\in S,$$

under very weak assumptions concerning the domain of the functions F,G,H[30]. Namely, we assume that S is a Tabor groupoid, i.e., it is a set with an operation $\cdot$ such that

$$\forall_{x,y\in S}\,\exists_{k\in\mathbb{N}}\,(xy)^{2^k}=x^{2^k}y^{2^k}.$$

The powers of the form x^{2^k} are defined recursively:

$$x^{2^0}=x,\quad x^{2^{k+1}}=x^{2^k}x^{2^k}.$$

Theorem 9.1. *Let S be a Tabor groupoid with left- and right neutral element. Suppose that V is symmetric, bounded ideally convex*[31] *subset of the Banach space E. Let $f,g,h\colon S\to E$ satisfy the condition*

$$f(xy)-g(x)-h(y)\in V,\quad x,y\in S.$$

Then there exist functions $F,G,H\colon S\to E$ such that the Pexider functional equation is satisfied

$$F(xy)=G(x)+H(y),\quad x,y\in S,$$

[30]Stability of the Pexider functional equation was investigated also in some other papers: [91], [39], [68], but in these papers the assumptions concerning the domain were stronger, however the assumptions concerning the target space were weaker than in our paper [II].

[31]The subset V of Banach space E is ideally convex [75], if for every bounded sequence $d_1,d_2,\ldots\in V$ and every sequence $\alpha_1,\alpha_2,\ldots\ge 0$ such that $\sum_{k=1}^{\infty}\alpha_k=1$, we have $\sum_{k=1}^{\infty}\alpha_k d_k\in V$.

and

$$F(x) - f(x) \in 3V, \; G(x) - g(x) \in 4V, \; H(x) - h(x) \in 4V, \quad x \in S.$$

9.2 The results from [VIII]

In [VIII] we present some results concerning Tabor groupoids. Among others: conditions sufficient for a semigroup with an idempotent element to be a Tabor groupoid, an example of a group which is not a Tabor groupoid and an example of nontrivial Tabor groupoid[32]. Moreover we formulated the following corollary from an earlier paper of P. Volkmann:

Theorem 9.2. *Let S be a Tabor groupoid, V bounded closed and convex subset of a Banach space E, and $f\colon S \to E$ satisfy*

$$f(xy) - f(x) - f(y) \in V, \quad x, y \in S.$$

Then there exists a unique additive function $a\colon S \to E$ such that

$$a(x) - f(x) \in V, \quad x \in S.$$

The following characterization of bounded perturbation of additive functions was proved:

Theorem 9.3. *Let S be a Tabor groupoid, A bounded and closed subset of a Banach space E, $f\colon S \to E$. Then the following conditions are equivalent:*

(P) $f = a + r$, where $a\colon S \to E$ is additive, $r(x) \in A$ for $x \in S$;

(Q) there exist bounded sets $B, C \subset E$ such that

$$f(xy) - f(x) - f(y) \in B, \quad x, y \in S,$$

$$2^k f(x) - f(x^{2^k}) \in 2^k A + C, \quad x \in S, k \in \mathbb{N}.$$

In this paper some stability results concerning the functional equation (6.12) were included, which was already mentioned in the Chapter 6 of this work.

9.3 Results from [IX]

In [IX] we prove the theorem concerning the stability of the following version of Pexider equation:

$$f(xy) = g(x)h(y) + k(y),$$

for functions defined on an amenable semigroup.

Theorem 9.4. *Let S be an amenable semigroup with a neutral element, $f, g, h, k\colon S \to \mathbb{C}$, $\varepsilon \geq 0$,*

$$|f(xy) - g(x)h(y) - k(y)| \leq \varepsilon, \quad x, y \in S.$$

[32] Some other results concerning Tabor groupoids can be found in [114].

Then there exist functions $F, G, H, K\colon S \to \mathbb{C}$ satisfying equation:

$$F(xy) = G(x)H(y) + K(y), \quad x, y \in S,$$

such that the differences $f - F$, $g - G$, $h - H$ and $k - K$ are bounded.

In the proof we use the invariant mean method which is a popular method for proving stability of functional equations.

9.4 The results from [XII]

In [XII] we prove the following abstract version of Hyers Theorem:

Theorem 9.5. *Let Y be an abelian group divided in a unique way by 2, let $B \subset Y$ be a $\frac{1}{2}$-convex set such that*

$$\bigcap_{n\in\mathbb{N}} \frac{1}{2^n}(B - B) = \{0\}.$$

Suppose that for every sequence $(y_n)_{n\in\mathbb{N}_0}$ of point of group Y the following implication holds true:
if

$$y_{n+1} + \frac{1}{2^{n+1}}B \subseteq y_n + \frac{1}{2^n}B, \quad n \in \mathbb{N}_0,$$

then

$$\bigcap_{n\in\mathbb{N}_0} (y_n + \frac{1}{2^n}B) \neq \emptyset.$$

Then for an arbitrary abelian semigroup $(S, +)$ and an arbitrary function $f\colon S \to Y$ satisfying

$$f(s + t) - f(s) - f(t) \in B, \quad s, t \in S,$$

there exists a unique additive function $a\colon S \to Y$ such that

$$a(s) - f(s) \in B, \quad s \in S.$$

The motivation was to explain why in different "theorems of Hyers' type" there are assumptions either about the completness of the target space or about the compactness of the "set of errors" B. It turns out that it is a consequence of intersection properties (Cantor Theorem for complete spaces and the finite intersection property of a family of compact sets). Two corollaries for maps with values in topological spaces – either with the assumption of sequentiall completness of Y, or compactness of B – generalized many of the earlier results (a series of papers with the versions of Hyers Theorem for maps in some particular complete spaces – for example "2-Banach spaces" [93], complete nonarchimedean spaces [81], "β-Banach spaces" [97] – or version of Hyers Theorem with compact "error set" B [XII, Theorem 1.2], [4]).

Under the assumption of completeness of Y we have:

Corollary 9.1. *Let $B \neq \emptyset$ be a $\frac{1}{2}$-convex closed and bounded subset of abelian, uniquely divisible by 2, sequentially complete topological Hausdorff group Y, without elements of finite order. Then for an arbitrary abelian semigroup $(S,+)$ and an arbitrary function $f\colon S \to Y$ fulfilling*

$$f(s+t) - f(s) - f(t) \in B, \quad s,t \in S,$$

there exists a unique additive function $a\colon S \to Y$ such that

$$a(s) - f(s) \in B, \quad s \in S.$$

In the case of compactness of "error set" B we have:

Corollary 9.2. *Let Y be an abelian, uniquely divisible by 2, topological group such that the map*

$$Y \ni x \mapsto \frac{1}{2}x \in Y$$

is continouos and let B be $\frac{1}{2}$-convex and compact subset of Y fulfilling

$$\bigcap_{n=0}^{\infty} \frac{1}{2^n}(B-B) = \{0\}.$$

Then for an arbitrary abelian semigroup $(S,+)$ and an arbitrary function $f\colon S \to Y$ satisfying

$$f(s+t) - f(s) - f(t) \in B, \quad s,t \in S,$$

there exists a unique additive function $a\colon S \to Y$ such that

$$a(s) - f(s) \in B, \quad s \in S.$$

The list of publications containing results from Ph. D. Thesis:

[XIV] Barbara Przebieracz, *Approximately iterable functions*, Proceedings of ECIT06, Grazer Math. Ber., Bericht Nr 351 (2007), 139–157.

[XV] Barbara Przebieracz, *The closure of the set of iterable functions*, Aequationes Math. 75 (2008), 239–250.

[XVI] Barbara Przebieracz, *Weak almost iterability*, Real Analysis Exchange 34(2), (2008/2009), 359–376.

10. Results included in Ph. D. Thesis

In [120] M.C. Zdun characterized continuous selfmappings of an interval $X \subset \mathbb{R}$, which are embeddable in continuous iteration semigroups, i.e., these continuous $f\colon X \to X$, for which there exists a continuous solution $F\colon (0,\infty) \times X \to X$ of the translation equation

$$F(s, F(t,x)) = F(t+s, x), \quad s,t \in (0,\infty),\ x \in X,$$

satisfying

$$f(x) = F(1,x), \quad x \in X.$$

Functions embeddable into continuous iteration semigroups are called *iterable*. As a response to the problem posed by E. Jen in monograph of Gy. Targonski [113, problem (3.1.12)], W. Jarczyk proposed the following definition of functions which are, in a sense, close to the iterable ones:

A continuous function $f\colon X \to X$ is called *almost iterable*, if there exists an iterable function $g\colon X \to X$ such that

$$\lim_{n\to\infty} (f^n(x) - g^n(x)) = 0, \quad x \in X, \tag{10.1}$$

and this convergence is uniform on every component of the set[33] $[a_f, b_f] \setminus \mathrm{Per}(f,1)$.

In [51] one can find a characterization of almost iterability.

In [XVI] I investigated some generalizations of almost iterability, namely the form of the functions satisfying (10.1) without the additional assumption about uniform convergence which appeared in the definition of almost iterability (such functions were called *weak almost iterable*), furthermore, I described functions satisfying the condition (10.1) for x from some "large" set either in a sense of measure or topology. The following theorem contains the characterization of weak almost iterability.

[33] Let me remind that $\mathrm{Per}(f,1)$ stands for the set of all fixed points of function f and $\mathrm{Per}(f,2)$ stands for all the periodic points of order 2 of function f, i.e.,

$$\mathrm{Per}(f,1) = \{x \in X : f(x) = x\}, \quad \mathrm{Per}(f,2) = \{x \in X : f^2(x) = x,\ f(x) \neq x\},$$

moreover a_f and b_f denotes the smallest and the greatest, respectively, fixed point of f.

Theorem 10.1. *A continuous function $f\colon X \to X$ is weak almost iterable if and only if* $\mathrm{Per}(f,2) = \emptyset$ *and there are points $a_i, b_i \in \mathrm{Per}(f,1)$, $i \in I$, such that $[a_f, b_f] = \bigcup_{i \in I}[a_i, b_i]$ and for every $i \in I$ one of the following possibilities holds true:*

(i) $a_i = b_i$;

(ii) $(a_i, b_i) \cap \mathrm{Per}(f,1) = \emptyset$, $f([a_i, b_i]) = [a_i, b_i]$;

(iii) $(a_i, b_i) \cap \mathrm{Per}(f,1) = \{c_i\}$, $x < f(x) < b_i$, *for* $x \in (a_i, c_i)$, $a_i < f(x) < x$ *for* $x \in (c_i, b_i)$;

(iv) $(a_i, b_i) \cap \mathrm{Per}(f,1) = \emptyset$, $b_i = b_f$, $f(x) > x$ *for* $x \in (a_i, b_i)$;

(v) $(a_i, b_i) \cap \mathrm{Per}(f,1) = \emptyset$, $a_i = a_f$, $f(x) < x$ *for* $x \in (a_i, b_i)$.

In [XIV] I proposed another definition of functions “close to” the iterable ones. The main result of this paper is included in the following theorem:

Theorem 10.2. *A continuous function $f\colon X \to X$ satisfies the condition*

(W) *for every $\varepsilon > 0$ there exists an iterable function $g\colon X \to X$ and a positive integer n_0 such that $|f^n(x) - g^n(x)| < \varepsilon$ for $n \geq n_0$, $x \in X$,*

if and only if

(i) *the restriction $f|_{[a_f, b_f]}$ is an increasing function*

and one of the following conditions holds: (ii)–(iv)

(ii) $\mathrm{Per}(f,2) = \emptyset$,

(iii) $\bigcap_{n \in \mathbb{N}} f^n(X) = [a_f, b_f]$,

(iv) *for every $x \in X$ the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges.*

The conditions listed below, equivalent to belonging to the closure of the set of iterable functions come from [XV].

Theorem 10.3. *Let $f\colon X \to X$ be a continuous function. Then the following conditions are equivalent:*

(1) *There exist points $x_1, x_2 \in X$ such that*

$$x_1 \leq f(x_1) \leq f(x) \leq f(x_2) \leq x_2, \quad x \in X,$$

and the restriction $f|_{[x_1, x_2]}$ increases.

(2) *For every $\varepsilon > 0$ there exists an iterable function $g\colon X \to X$ such that*

$$|f^n(x) - g^n(x)| < \varepsilon, \quad n \in \mathbb{N}, x \in X.$$

(3) *For every $\varepsilon > 0$ there exists an iterable function $g\colon X \to X$ such that $|f(x) - g(x)| < \varepsilon$ for $x \in X$.*

(4) *For every* $\varepsilon > 0$ *there exists an* ε*-iteration semigroup*[34] $F\colon (0,\infty) \times X \to X$ *such that*

$$|f^n(x) - F(n,x)| < \varepsilon, \quad n \in \mathbb{N},\, x \in X.$$

(5) *For every* $\varepsilon > 0$ *there exists an* ε*-iteration semigroup* $F\colon (0,\infty) \times X \to X$ *such that*

$$|f(x) - F(1,x)| < \varepsilon, \quad x \in X.$$

[34] We say that a continuous function $F\colon (0,\infty) \times X \to X$ is an ε-iteration semigroup, if it satisfies

$$|F(s, F(t,x)) - F(t+s,x)| \le \varepsilon, \quad \text{for } \ x \in X,\ s,t \in (0,\infty).$$

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