Summary of Professional Accomplishments

Hanna Wojewódka-Ściążko

September 28, 2023

Contents

1 Name

Hanna Wojewódka-Ściążko

2 Diplomas and degrees

2015

Doctor of Mathematical Sciences in the field of mathematics
(doctoral thesis defended with distinction)

The degree was conferred by a resolution of the Council of the Faculty of Mathematics, Physics and Informatics of the University of Gdańsk, based on the doctoral dissertation entitled *Ergodyczne własności pewnych stochastycznych układów dynamicznych* (*Ergodic properties of certain stochastic dynamical systems*), prepared under the supervision of prof. dr hab. Tomasz Jakub Szarek.

2011

Master of Mathematics

The title was awarded at the Faculty of Mathematics, Physics and Informatics of the University of Gdańsk, based on the thesis entitled *Modele matematyki finansowej o dynamicznej strukturze czasowej* (*Models of financial mathematics with a dynamic time structure*), written under the supervision of dr hab. Henryk Leszczyński, prof. UG.

3 Information on employment in research institutes or faculties/departments

2016–Present

Assistant Professor (Adiunkt)
at the Institute of Mathematics, University of Silesia in Katowice

(Full-time position)

(1 October 2021 – 30 June 2023: Unpaid leave; 2 June 2020 – 31 May 2021: Maternity and parental leaves; 2 March 2020 – 1 June 2020: Inability to work)

2019–2023

Postdoctoral Research Fellow (Adiunkt)
Institute of Theoretical and Applied Informatics, Polish Academy of Sciences

(1 December 2019 – 30 September 2021: 50% position; 1 October 2021 – 30 June 2023: Full-time position; 1 July 2023 – 31 October 2023: 50% position)

(2 June 2020 – 31 May 2021: Maternity and parental leaves; 2 March 2020 – 1 June 2020: Inability to work)

2015–2016

Lecturer
at the Faculty of Applied Physics and Mathematics, Gdańsk University of Technology

(25% position)

2015–2016

Postdoctoral Research Fellow (Adiunkt)
at the Institute of Theoretical Physics and Astrophysics, University of Gdańsk, with a delegation to work at the National Quantum Information Centre in Gdańsk
(1 October 2015 – 16 December 2015: 50% position; 17 December 2015 – 30 September 2016: 75% position)

2013–2015

Assistant
at the Institute of Theoretical Physics and Astrophysics, University of Gdańsk, with a delegation to work at the National Quantum Information Centre in Gdańsk
(50% position)

4 Description of the achievements, set out in art. 219 para 1. point 2 of the Act (and other, thematically related, results of the applicant)

Title of the scientific achievement

Ergodic description of certain classes of non-stationary Markov processes evolving in Polish spaces

4.1 List of publications constituting the scientific achievement

[H1] R. Kukulski, H. Wojewódka-Ściążko, *The e-property of asymptotically stable Markov-Feller operators*, Colloq. Math. **165** (2021), 269–283, DOI 10.4064/cm8165-6-2020.

[H2] D. Czapla, K. Horbacz, H. Wojewódka-Ściążko, *Ergodic properties of some piecewise deterministic Markov process with application to gene expression modelling*, Stoch. Proc. Appl. **130** (2020), no. 5, 2851–2885, DOI 10.1016/j.spa.2019.08.006.

[H3] D. Czapla, S.C. Hille, K. Horbacz, H. Wojewódka-Ściążko, *Continuous dependence of an invariant measure on the jump rate of a piecewise-deterministic Markov process*, Math. Biosci. Eng. **17** (2020), no. 2, 1059–1073, DOI 10.3934/mbe.2020056.

[H4] D. Czapla, K. Horbacz, H. Wojewódka-Ściążko, *A useful version of the central limit theorem for a general class of Markov chains*, J. Math. Anal. Appl. **484** (2020), no. 1, 123725, DOI 10.1016/j.jmaa.2019.123725.

[H5] D. Czapla, K. Horbacz, H. Wojewódka-Ściążko, *The Strassen invariance principle for certain non-stationary Markov-Feller chains*, Asymptot. Anal. **121** (2021), no. 1, 1–34, DOI 10.3233/ ASY-191592.

[H6] K. Czudek, T. Szarek, H. Wojewódka-Ściążko, *The law of the iterated logarithm for random interval homeomorphisms*, Isr. J. Math. **246** (2021), 47–53, DOI 10.1007/s11856-021-2235-9.

4.2 List of other scientific articles related to the topic of the scientific achievement

Latest result (paper under review)

[N1] R. Kukulski, H. Wojewódka-Ściążko, *The e-property of asymptotically stable Markov semigroups*, under review in Results Math., arXiv:2211.16424 [math.PR] (2022), DOI 10.48550/arXiv.2211.16424.

Articles describing ergodic properties of Markov processes with values in Polish spaces, published after the conferment of the PhD degree

[E1] D. Czapla, K. Horbacz, H. Wojewódka-Ściążko, *The central limit theorem for Markov processes that are exponentially ergodic in the bounded-Lipschitz norm*, Qual. Theory Dyn. Syst. **23** (2023), no. 7, DOI 10.1007/s12346-023-00862-4.

[E2] D. Czapla, K. Horbacz, H. Wojewódka-Ściążko, *Exponential ergodicity in the bounded-Lipschitz distance for some piecewise-deterministic Markov processes with random switching between flows*, Nonlinear Anal. **213** (2022), no. 112678, DOI 10.1016/j.na.2021.112678.

[E3] D. Czapla, K. Horbacz, H. Wojewódka-Ściążko, *On absolute continuity of invariant measures associated with a piecewise-deterministic Markov process with random switching between flows*, Nonlinear Anal. **13** (2021), no. 112522, DOI 10.1016/j.na.2021.112522.

[E4] D. Czapla, S.C. Hille, K. Horbacz, H. Wojewódka-Ściążko, *The law of the iterated logarithm for a piecewise deterministic Markov process assured by the properties of the Markov chain given by its post-jump locations*, Stoch. Anal. Appl. **39** (2021), no. 2, 357–379, DOI 10.1080/07362994.2020.1798252.

The results on ergodic properties of certain Markov dynamical systems, covering, e.g., a simple cell division model, obtained during doctoral studies

[D1] S.C. Hille, K. Horbacz, T. Szarek, H. Wojewódka, *Law of the iterated logarithm for some Markov operators*, Asymptot. Anal. **97** (2016), no. 1–2, 91–112, DOI 10.3233/ASY-151344.

[D2] S.C. Hille, K. Horbacz, T. Szarek, H. Wojewódka, *Limit theorems for some Markov chains*, J. Math. Anal. Appl. **443** (2016), no. 1, 385–408, DOI 10.1016/j.jmaa.2016.05.022.

[D3] H. Wojewódka, *Exponential rate of convergence for some Markov operators*, Statist. Probab. Lett. **83** (2013), no. 10, 2337–2347, DOI 10.1016/j.spl.2013.05.035.

4.3 Description of the achievements

4.3.1 Introduction

Markov operators* acting on measures and families of these operators forming semigroups (the so-called *Markov semigroups*) derive naturally from *Markov processes (discrete-time and continuous-time, respectively). They are employed to describe the dynamics of the distributions of these processes.

Random Markov dynamical systems, characterized either by chains or continuous-time processes, in particular *piecewise deterministic Markov processes* (PDMPs), have numerous applications, among others, in the theory of iterated function systems (IFSs) or (semi)fractals (see, e.g., [BD85, BDEG88, Las95, LM98, DF99, KS20]), theory of partial differential equations (see [Hai02, LS06, HM08, KPS10, Sza13], just to name a few, and also [SSU10, KPS13, KS14], which all refer to passive tracer models), storage modelling [BKKP05], internet traffic [GR09] or biology – as stochastic models describing the dynamics of molecular biology, such as gene expression and autoregulation [CDMR12, MTKY13, HHS16], cell division [LM99], excitable membranes [RTW12] or population dynamics [AHVG15, BL16, RTK17].

So far my research interests have focused mainly on certain *ergodic properties* of Markov dynamical systems (including the existence and uniqueness of invariant probability measures, equicontinuity, asymptotic stability, limit theorems and others). I have contributed to the development of the theory, especially in the context of considering general metric state spaces – specifically *Polish* (that is, complete and separable) metric state spaces, which are not necessarily locally compact. Notably, these results apply to infinite-dimensional systems.

The transition from (locally) compact spaces to Polish spaces is not a straightforward generalization (note that balls in such spaces do not need to be compact). As emphasized by prominent mathematicians such as M. Hairer [Hai02, HM08, CH15] or A. Lasota [Las95], just to name a few, it rather represents a significant qualitative advance. **In Polish spaces, most of the methods developed for locally compact and σ-compact spaces become impossible to apply, which makes it necessary to develop new concepts.**

Additionally, such a research is strongly motivated by specific applications (see, e.g., [HHS16], which provides an example from molecular biology indicating the importance of considering a non-locally compact space as the state space in the abstract framework; [EHM15], where mild solutions to a measure-valued mass evolution problem with flux boundary conditions are established; or [Pic19], elaborating on measure differential equations).

Below we present a concise summary of the results of the series *Ergodic description of certain classes of non-stationary Markov processes evolving in Polish spaces.*

[H1] The relations between *asymptotic stability* and the *e-property* of Markov operators acting on measures defined on general (Polish) metric spaces are studied. While usually much attention is paid to asymptotic stability (and the e-property has been for years verified only to establish it), it should be noted that the e-property itself is also important as it, e.g., ensures that numerical errors in simulations are negligible. Here, we prove that any asymptotically stable Markov–Feller operator has the e-property everywhere outside a set of first category. We also provide an example showing that this result is tight. Moreover, an equivalent criterion for the e-property is proposed.

[H2] We investigate a PDMP with a Polish state space, whose deterministic behaviour between random jumps is governed by a finite number of semiflows, and any state right after the jump is attained by a randomly selected continuous transformation. It is assumed that the jumps appear at random moments, which coincide with the jump times of a Poisson process with intensity λ. We provide tractable conditions ensuring a form of *geometric ergodicity* and the *strong law of large numbers* (SLLN) for the chain given by the post-jump locations. Further, we establish a *one-to-one correspondence between invariant probability measures* of the chain and those of the PDMP. These results enable us to derive the SLLN for the latter. The studied dynamical system is inspired by certain models of gene expression and autoregulation.

[H3] We study the PDMP introduced in [H2] (albeit with just one semiflow, determining the behavior of the system between random jumps). The aim of the paper is to prove the *continuous dependence of the unique invariant probability measure of this PDMP on the jump rate* λ. While limit theorems provide the theoretical foundation for successful approximation of the invariant measures, this result asserts the stability of this procedure, at least locally in parameter space. Moreover, it confirms the suitability of using this model as a tool to analyse the stochastic dynamics of gene expression in prokaryotes.

[H4] In the paper we propose certain conditions, relatively easy to verify, which ensure the *central limit theorem* (CLT) for some general class of non-stationary Markov-Feller chains (with values in Polish metric spaces). This class may be briefly specified by the following two properties: firstly – the transition operator of the chain under consideration enjoys a non-linear Lyapunov-type condition, and secondly – there exists an appropriate Markovian coupling whose transition probability function can be decomposed into two parts, one of which is contractive and dominant in some sense. The given conditions guarantee both the *geometric ergodicity (in the Fortet-Mourier metric)* and the CLT. To justify the usefulness of our criterion, we further verify it for a particular discrete-time Markov dynamical system (introduced in [H2]).

[H5] We propose certain conditions implying the functional *law of the iterated logarithm* (LIL), the so-called *Strassen invariance principle for the LIL*, for some general class of non-stationary Markov-Feller chains (specified as above) with values in Polish metric spaces. In the final part of the paper we also present an example application of our main theorem to the specific mathematical model describing stochastic dynamics of gene expression (introduced in [H2]).

[H6] We prove the LIL for non-stationary Markov chains generated by *IFSs consisting of orientation-preserving homeomorphisms of the interval.* The result enriches the ergodic decription of this class of Markov chains, previously studied by K. Czudek and T. Szarek in terms of ergodicity and the CLT [Isr. J. Math., 239:75–291, 2012]. Note, however, that the chains considered here may not converge exponentially to their equilibria. Therefore, the techniques developed in [H5] (or any similar results) do not apply.

A more detailed discussion of these results (as well as those presented in the articles [N1] and [E1]–[E4], all thematically related to the series *Ergodic description of certain classes of non-stationary Markov processes evolving in Polish spaces*), which includes the historical background of the research, references to other scientific works, a description of the proof techniques used and the presentation of the remaining open questions (some of which we plan to address in the future), can be found in the next section.

4.3.2 Overview of the results

A brief historical background of this part of research on the asymptotics of Markov processes that uses equicontinuity properties

The asymptotic behavior of Markov processes, including the existence and uniqueness of their *stationary distributions*, as well as the weak convergence of their laws to unique stationaries that is independent of their initial distributions (*asymptotic stability*), has been widely studied over the years. In this connection, various techniques, in particular those referring to *equicontinuity properties* of families of Markov operators, have been introduced (see, e.g., [Ste94, LS06, Wor10, SW11, Cza12, CH14, WW18], where certain ergodic properties of Markov chains are established, or [SSU10, KS11], where the asymptotic behavior of continuous-time Markov processes is studied).

Initially, these type of methods have been developed for Markov processes evolving in compact metric spaces (cf. [Jam64]) or locally compact Hausdorff topological spaces (cf. [MT93b]). Further, the so-called *lower-bound technique* for equicontinuous families of Markov operators has been introduced to prove results for processes evolving in general (Polish) metric spaces (see, e.g., [LS06, Sza13]; cf. [Sza03, Sza06], where one the first results concerning asymptotics of Markov operators evolving in Polish metric spaces are obtained). In the literature the concepts like the *e-property* [SW11, Cza12, CH14], the *eventual e-property* [Wor10, Cza18], *the Cesáro e-property* [Wor10] or even *uniform equicontinuity on balls* [HHS16] have been considered. Recently, by utilizing a *Schur-like property* for measures, the authors of [HSWZ21] have also established a rigorous connection between the concepts of equicontinuity for Markov operators acting on measures and their dual operators acting on functions.

Relations between asymptotic satability and the e-property

Let (E, ρ) be a given Polish metric space, and let R be a sufficiently large subset of the set $\mathrm{C}_b(E)$ consisting of all real-valued bounded continuous functions on E. We say that a regular E-valued Markov operator P (with the dual operator denoted by the same symbol) has the *e-property in* R if the family $\{P^n f\}_{n\in\mathbb{N}_0}$ of iterates is equicontinuous for all $f \in \mathrm{R}$, that is, for all $f \in \mathrm{R}$

$$\lim_{x\to z} \sup_{n\in\mathbb{N}_0} |P^n f(x) - P^n f(z)| = 0 \quad \text{for every} \quad z \in E.$$

Similarly, a regular Markov semigroup $\{P(t)\}_{t\in\mathbb{R}_+}$ has the *e-property in* R if the family $\{P(t)f\}_{t\in\mathbb{R}_+}$ is equicontinuous for all $f \in \mathrm{R}$. In many papers (see [KPS10, Wor10, SW11, HSZ17, Cza18], just to name a few), R is assumed to be the set $\mathrm{Lip}_b(E)$ of all bounded Lipschitz functions, although it can also be the set of bounded continuous functions with bounded (or compact) support (cf., e.g., [MT93b, Ste94, Cza12]), or even the entire set of $\mathrm{C}_b(E)$ (as in our papers [H1] and [N1]). Definitions of the e-property for different sets R are generally non-equivalent. However, assuming that a regular Markov operator P is asymptotically stable, we can use some of these notions interchangebly (see Remark 2.1 and Lemma 3.4 in [H1]). The comparison of the e-property notions for different sets R in the case of Markov semigroups is presented in Appendix II in [N1]. In that case, not only asymptotic stability of $\{P(t)\}_{t\in\mathbb{R}_+}$ but also its *stochastic continuity at zero* (defined as in [Dyn65, Dyn00, KW12]) are needed to make some of these notions equivalent.

Knowing the criteria on asymptotic stability of Markov operators with the e-property, one may ask about a reverse relation, namely: *Does asymptotic stability immediately imply* *the e-property?*. The answer is negative and it is given in [HSZ17]. More precisely, the authors of [HSZ17] have provided examples of Markov-Feller operators which are asymptotically stable, but which do not have the e-property. Simultaneously, they have proved that any asymptotically stable Markov-Feller operator with an invariant probability measure μ_* such that the interior of its support is non-empty ($\texttt{Int}(\text{supp}(\mu_*)) \neq \emptyset$) satisfies the e-property (see [HSZ17, Theorem 2.3]).

In [H1], we generalize [HSZ17, Theorem 2.3], i.e., we prove that any asymptotically stable Markov–Feller operator has the e-property off a set of first category (see Theorem 3.1 in [H1]; cf. Section 4.3.3 of this Summary). Moreover, in Theorem 3.5 in [H1], we propose an equivalent condition for the e-property of asymptotically stable Markov–Feller operators. Namely, **we prove that an asymptotically stable Markov–Feller operator has the e-property if and only if it has the e-property at least at one point of the support of its invariant probability measure**. These two results then naturally imply [HSZ17, Theorem 2.3] (cf. Section 4.3.3 of this Summary). In fact, they imply even more, because thanks to Lemma 3.4 in [H1] we know that the assumptions made lead to e-property in $\mathrm{C}_b(E)$ (the assertion of [HSZ17, Theorem 2.3] was originally formulated for the e-property in $\mathrm{Lip}_b(E)$, which is less general). Summary of [H1].

We also provide two examples in [H1]. In the first of them, **we define an asymptotically stable Markov–Feller operator such that the set of points at which it fails the e-property is dense**. The example thus shows that the main result of [H1] is tight. In the second example, **we construct an asymptotically stable Markov–Feller operator for which the set of points without the e-property is uncountable**. For details see Section 4 in [H1].

The issue of the relationship between the e-property and asymptotic stability is also addressed in our very last paper [N1], where we answer the following question: *When does an asymptotically stable Markov semigroup have the e-property?*. **One of the main theorems of that paper (Theorem 1) is an equivalent of [HSZ17, Theorem 2.3] for Markov semigroups**, and so its proof is based on certain ideas derived from [HSZ17]. The difference is that conditions guaranteeing the e-property for a single asymptotically stable Markov operator P (that is, the Feller property and the non-emptiness of the interior of the support of its invariant probability measure) are insuffcient to guarantee it for an asymptotically stable Markov semigroup $\{P(t)\}_{t\in\mathbb{R}_+}$ (counter-examples are provided in Section 2.2 of [N1]). Under these conditions, only the *eventual e-property* in $\mathrm{C}_b(E)$, that is, the e-property in $\mathrm{C}_b(E)$ that holds from a certain point in time, rather than over the entire time interval $t \in \mathbb{R}_+$ (for a precise definition see [Cza18, Eq. (5.11)] or Section 1.3 in [N1]), can be established. To prove the e-property, an additional assumption of *strong stochastic continuity* (defined as in [EK86, p. 6]) has to be made. More precisely, **we demonstrate that any Markov semigroup that is strongly stochastically continuous, and possesses the eventual e-property, also has the e-property** (Theorem 2 in [N1]). Interestingly, the assumption cannot be weakened. **Example 2 in [N1] illustrates that stochastic continuity in its weaker form, that is, with pointwise convergence in the place of convergence in the supremum norm, does not necessarily imply the desired assertion, unless the underlying phase space is compact** (cf. Appendix I in [N1]). As a corrolary of the two above mentioned theorems (i.e., Theorems 1 and 2 in [N1]), **we obtain that an asymptotically stable Markov-Feller semigroup possesses the e-property if it is also strongly stochastically continuous and the interior of the support of its invariant probability measure is non-empty** (Corollary 1 in [N1]). Summary of [N1].

A slightly more general result, assuming 'eventual continuity' instead of asymptotic stability, was later established in [LL23] (although there the authors adhere to the stronger definition of the e-property, namely the e-property in $\mathrm{Lip}_b(E)$). In contrast, we focus on providing a justification for the tightness of the results presented in this [N1], while also addressing the fifth open question stated in [LL23]. This question pertains to the possibility of weakening the assumption of strong stochastic continuity of Markov-Feller semigroups to stochastic continuity while still preserving the equivalence between their e-property and eventual e-property.

Finally, it is worth noting that stochastic continuity is not a highly restrictive requirement when working with Markov semigroups $\{P(t)\}_{t\in\mathbb{R}_+}$. It is assumed, for instance, to ensure joint measurability of a Markov semigroup (as stated in [Wor10, Proposition 3.4.5]) or to uniquely characterize a Markov-Feller semigroup through its (weak) infinitesimal operator (see [Dyn65, Theorem 2.5]).

Asymptotic stability, especially if achieved with the exponential rate, is one of the most desired ergodic properties of Markov processes, but it is the e-property that, if additionally met, guarantees that certain numerical errors can be treated as negligible in simulations (cf. also [CH14], where it is proven that an asymptotically stable Markov-Feller operator converges to its stationary uniformly, provided that it satisfies a form of equicontinuity condition). **This indicates that theoretical results established in [H1] and [N1] are important even from the point of view of applications.**

Further research related to this topic should focus on replacing the assumption of the non-emptiness of the interior of the support of the invariant probability measure of a given Markov operator/semigroup with another condition that is less restrictive and hopefully easier to verify, such as the assumption that the unique invariant probability measure is atomless.

Asymptotic coupling techniques

As it has already been mentioned in Section 4.3.2: *A brief historical background of this part of research on the asymptotics of Markov processes that uses equicontinuity properties*, asymptotic behaviour of Markov dynamical systems has been often examined using techniques that exploit various equicontinuity properties. However, to prove *ergodicity* of a given random system and, additionally, even estimate its rate, a quite different concept, based on the application of *asymptotic coupling*, can be applied. This modern approach has been introduced by M. Hairer in his prominent paper [Hai02], inspired mainly by the article [Mat02] of J.C. Mattingly on the 2nd Navier-Stokes equation. Further, the method has been succesfully developed for both Markov operators, in particluar, those describing the evolution of IFSs (cf. [Ś11, Cza18, CK19, KS20] or [D3], [H2]), and Markov semigroups, chracterizing, among others, PDMPs (see, e.g., [CH15] or [E2]), all evolving in general (Polish) metric spaces.

The underlying idea of all asymptotic coupling techniques can be briefly described as follows. For a given Markov chain $\{\phi_n\}_{n\in\mathbb{N}_0}$ with transition law P (note that, according to the convention employed, e.g., in [MT93b, OLK12], the corresponding Markov operator acting on measures, as well as its dual acting on functions, are also denoted by the same symbol P) we consider two instances of it: one with initial point x_0, denoted by $\{\phi_n^{(1)}\}_{n\in\mathbb{N}_0}$, and the other one with initial point y_0, denoted by $\{\phi_n^{(2)}\}_{n\in\mathbb{N}_0}$. The goal is to bring the trajectories of $\{\phi_n^{(1)}\}_{n\in\mathbb{N}_0}$ and $\{\phi_n^{(2)}\}_{n\in\mathbb{N}_0}$ as close as possible by creating appropriate correlations between these chains. Under some (fairly general) assumptions, this can be achieved by decomposing the transition law of the coupled chain $\{(\phi_n^{(1)},\phi_n^{(2)})\}_{n\in\mathbb{N}_0}$, the so called *Markovian coupling*, into two parts, one of which is constructive and dominant in some sense (details can be found, e.g., in [H4]).

In the research literature one can find the notion of a *coupling time* τ_{couple}, which is the random moment at which both copies of the Markov chain $\{\phi_n\}_{n\in\mathbb{N}_0}$ reach the same state for the first time, i.e. $\tau_{\text{couple}} = \min\{n \in \mathbb{N}_0 : \phi_n^{(1)} = \phi_n^{(2)}\}$. We know that if $\tau_{\text{couple}} < \infty$, then the so-called *successful coupling*, making chains stay together all the time, can be constructed (see, e.g., [MT93b, Lin02]). This is, however, not the case here, and thus, following M. Hairer [Hai02], we couple asymptotically the whole trajectories of Markov chains, which cannot meet at some finite time (cf. [HM08], which provides an example of a Markov chain evolving in an infinite dimensional space, whose two copies, starting from different initial points, induce mutually singular measures).

Finally, it is worth mentioning that adaptation of asymptotic coupling techniques can serve not only to establish exponential ergodicity but also limit theorems, including the SLLN (see [HS16]), the CLT (see [D2], [Hor16], [H4]), and the LIL (see [D1], [H5], [E4]). In particular, **the articles [H4] and [H5], both based on a creative adaptation of asymptotic coupling techniques, constitute an important contribution to the research on general versions of limit theorems for Markov processes with Polish state spaces.**

PDMPs driven by randomly switched semiflows

Our initial motivation for the creative use of asymptotic coupling techniques was to prepare an ergodic description of a random dynamical system, which serves, among others, to describe a simple model of cell division. This system has already been studied in terms of stability by A. Lasota and M.C. Mackey in [LM99]. Following the ideas of M. Hairer [Hai02] (cf. also [Ś11]), **we estimated the convergence rate to the unique stationary distribution of this system as geometric [D3], and subsequently established both the CLT [D2] and the LIL [D1]** (here, also referring to [MW00] and [HS73, BMS12], respectively). Simultaneously, R. Kapica and M. Ślęczka have adapted these techniques to establish their criterion ([KS20, Theorem 2.1]) on geometric ergodicity for Markov chains evolving in Polish spaces, with a particular application to random iterations with place dependant probabilities. All of the aforementioned results and underlying ideas proved to be useful in our study of a certain class of PDMPs with random switching between flows.

Summary of the articles [D1]–[D3].

PDMPs, introduced by M. Davis [Dav84] (see also [Cos90, Dav93, Dav06]), constitute a broad class of non-diffusive Markov processes, for which randomness arises solely from the *jump mechanism*, including jump times, post-jump locations, and other changes occurring at the moments of jumps. This extensive family of processes finds numerous applications in various fields such as biology [CDMR12, MTKY13, RTK17], storage modelling [BKKP05], internet traffic [GR09] or control and optimization [CD99, CD08]. PDMPs also appear as solutions of certain variations on the Poisson-driven stochastic differential equations (cf. [Hor02, MZ10, Kaz13, CK19]), mainly developed by A. Lasota and J. Traple [LT03], which have significant applications in biomathematics, physics and engineering (cf. [Sny75, DHT84]).

Here, we are concerned with some random dynamical system taking values in a Polish metric space Y (equipped with the Borel σ-algebra $\mathcal{B}(Y)$) **and perturbed by random jumps occurring at the jump times of a Poisson process.** This means that the span of time between consecutive jumps is exponentially distributed with a constant rate λ. Between any two adjacent jumps, the dynamics of these systems is driven by one of the semiflows, randomly selected from a finite set $\{S_i\}_{i\in I}$, where $S_i : \mathbb{R}_+ \times Y \to Y$, of all possible ones. The selection of the semiflow is governed by a matrix $[\pi_{ij}]_{i,j\in I}$ of continuous functions

The class of random dynamical systems studied in [H2]–[H5], [E1]–[E4].

$\pi_{ij} : Y \to [0,1]$, satisfying $\sum_{j\in I} \pi_{ij}(y) = 1$ for every $y \in Y$ and every $i \in I$. The state of the system right after a jump (commonly referred to as the *post-jump location*) depends randomly on the one immediately preceding it, and its probability distribution is governed by a given stochastic kernel $J : Y \times \mathcal{B}(Y) \to [0,1]$.

More specifically, we investigate a stochastic process $\Psi := \{(Y(t), \xi(t))\}_{t\in\mathbb{R}_+}$ with values in $X := Y \times I$, whose motion can be described as follows. Starting from some initial value (y_0, i_0), the process Ψ evolves deterministically, following $t \mapsto (S_{i_0}(t, y_0), i_0)$ until the first jump time, say $t_1 > 0$. At this moment the trajectory of the first coordinate jumps to another point of Y, say y_1, so that the probability that it will fall into $B \in \mathcal{B}(Y)$ is equal to $J(S_{i_0}(t_1, y_0), B)$. At the same time, the index of the 'active' semiflow, determined by $\{\xi(t)\}_{t\in\mathbb{R}_+}$, is randomly switched from i_0 to another (or the same) one i_1 with probability $\pi_{i_0 i_1}(y_1)$. Then the motion restarts from the new state (y_1, i_1) and proceeds as before. Formally, the process Ψ can be therefore defined by setting

$$Y(t) := S_{\xi_n}(t - \tau_n, Y_n) \quad \text{and} \quad \xi(t) := \xi_n \quad \text{for} \quad t \in [\tau_n, \tau_{n+1}), \ \ n \in \mathbb{N}_0, \tag{1}$$

where $\bar{\Phi} := \{(Y_n, \xi_n, \tau_n)\}_{n\in\mathbb{N}_0}$ is a time-homogeneous Markov chain with state space $X \times \mathbb{R}_+$ and transition law satisfying

$$\mathbb{P}\left(\bar{\Phi}_{n+1} \in \bar{A} \,|\, \bar{\Phi}_n = (y, i, s)\right) = \sum_{j\in I} \int_0^\infty \lambda e^{-\lambda t} \int_Y \mathbb{1}_{\bar{A}}(u, j, t+s)\, \pi_{ij}(u)\, J\left(S_i(t, y), du\right)\, dt \tag{2}$$

for any $n \in \mathbb{N}_0$ and any Borel set $\bar{A} \subset X \times \mathbb{R}_+$. Obviously, all the randomness of the PDMP Ψ is contained in the chain $\bar{\Phi}$. What is more, the sequence $\Phi := \{(Y_n, \xi_n)\}_{n\in\mathbb{N}_0}$ describing post-jump locations is itself an X-valued Markov chain (with respect to its natural filtration). Clearly, the transition law of this chain takes the form

$$\begin{aligned} P((y,i), A) &:= \mathbb{P}\left(\Phi_{n+1} \in A \,|\, \Phi_n = (y, i)\right) \\ &= \sum_{j\in I} \int_0^\infty \lambda e^{-\lambda t} \int_Y \mathbb{1}_A(u, j)\, \pi_{ij}(u)\, J\left(S_i(t, y), du\right)\, dt \end{aligned} \tag{3}$$

for any $n \in \mathbb{N}_0$ and any Borel set $A \subset X$.

The subclass of the PDMPs considered here somewhat resembles those investigated in [Cos90, CD99, CD08, BLBMZ12, BLBMZ15, BHS18, BS19]. **All the above-listed papers, however, focus on processes evolving in finite-dimensional (and thus locally compact) spaces. While proving the existence of invariant distributions and ergodicity (usually in the *total variation norm*) in such a setup, one can use various adaptations of conventional methods of S.P. Meyn and R.L. Tweedie [MT93a, MT93b], based mainly on the *Harris recurrence* (assured, e.g., by the *Hörmander-type bracket conditions*, just as in [BLBMZ15]) or some criteria referring to the so-called *drift towards a petite set*. These techniques, however, are mostly valid only for ψ-irreducible processes, which is, obviously, not the case in our framework (see also the relevant comments in the Introduction of** [H2] **and in the abstract of the paper [HM08] on, the so-called, *spectral gaps* in the *Wasserstein distance* for Markov semigroups on Banach spaces, where it is explained why a new approach is needed while working with infinite-dimensional spaces, in which the usual Harris or Doeblin conditions, geared toward total variation convergence, often fail to hold; cf. also [HM11]).** Comparison of PDMP classes studied here and in other works.

Let us indicate that in the series of articles [H2], [H3], [E1]–[E4] (focused on certain ergodic properties of Markov semigroups) we draw attention to a special case of the above-described PDMPs with the jump kernel J acting as a transition law of a randomly perturbed IFS, given by

$$J(y,B)=\int_{\mathrm{supp}(\nu)}\int_{\Theta}\mathbb{1}_B(w_\theta(y)+v)\,p_\theta(y)\,\vartheta(d\theta)\nu(dv)\quad\text{for each } y\in Y \text{ and any } B\in\mathcal{B}(Y), \tag{4}$$

in which case Y is a closed subset of a Banach space H, ν is a Borel probability measure on H with bounded support, Θ stands for an arbitrary topological space, endowed with a Borel measure ϑ, $\{w_\theta\}_{\theta\in\Theta}$ is a given family of continuous transformations from Y to itself such that $w_\theta(Y)+v\subset Y$ for any $v\in\mathrm{supp}(\nu)$, and $\Theta\ni\theta\mapsto p_\theta(y)\in\mathbb{R}_+$, $y\in Y$, are the associated state-dependent probability density functions with respect to ϑ.

In this setting, the model under consideration may serve as a framework for analysing the dynamics of gene expression in prokaryotes (see, e.g., [MTKY13, BTK16], Section 5.1 in [H2] or Example 7.3 in [E2]). **Moreover, if $\nu=\delta_0$, $\vartheta(\Theta)=1$ and $p_\theta\equiv 1$ for every $\theta\in\Theta$, then $\{Y(t)\}_{t\in\mathbb{R}_+}$ can be treated as the solution to a stochastic evolution equation with Poisson noise** (see, e.g., [Hor02, LT03, MZ10, Kaz13, CK19]).

Motivation to consider processes with Polish state spaces

The discrete-time dynamical system Φ with the jump kernel J defined as in (4) can also serve as a model for an autoregulated gene (as described in [HHS16] or in Section 5.2 of [H2]). **Thanks to this example, we see how important it is to consider non-locally compact spaces as state spaces in the abstract framework (in the gene autoregulation model, the phase space consists of continuous functions describing the concentration of chemical compounds at different points in the cell cytoplasm).** Its ergodicity and limit theorems are demonstrated in [H2], [H4] and [H5].

Ergodicity of the considered class of PDMPs and associated chains describing their post-jump locations

In recent years, much of our research has been dedicated to the study of the PDMP Ψ and the associated chain Φ which describes its post-jump locations (both introduced and roughly characterized in the previous section). Our main objective was to develop their ergodic descriptions, addressing the following questions:

- *Are these processes ergodic, and if so, how quickly do their distributions stabilize?*,
- *What are the properties of their invariant probability measures? How do they depend on model parameters? Are they singular or absolutely continuous with respect to the Lebesgue measure (in the case of a simplified model where the phase space of these processes is $\mathbb{R}^d$)?*,
- *Do any limit theorems hold for them?*.

Summary of [H2].

To begin our investigation, we first needed to ascertain the assumptions under which the processes Φ and Ψ possess stationary distributions (in fact, we have done even more at once, that is, **we have proposed relatively easily verifiable conditions under which these stationary distributions, say μ_*^Φ and μ_*^Ψ, respectively, do not only exist, but are also unique**).

For a specific version of the model with jump kernel J given by (4), in which case the transition law P of the chain Φ (cf. (3)) has the following form:

$$P((y,i),A) := \sum_{j\in I} \int_0^\infty \lambda e^{-\lambda t} \int_{\mathrm{supp}(\nu)} \int_\Theta \mathbb{1}_A\left(w_\theta\left(S_i(t,y)\right)+v,j\right)\, \pi_{ij}\left(w_\theta\left(S_i(t,y)\right)+v\right) \times p_\theta\left(S_i(t,y)\right)\,\vartheta(d\theta)\,\nu(dv)\,dt \quad \text{for any} \quad (y,i)\in X,\ A\in\mathcal{B}(X), \tag{5}$$

this has been done in [H2]: in Theorem 4.1 for the chain Φ and in Corollary 4.5 for the PDMP Ψ (cf. Section 4.3.3 of this Summary). **The proposed conditions are actually also sufficent to ensure *geometric ergodicty* of Φ** (see Theorem 4.1 in [H2]), which, roughly speaking, means that the unique invariant probability measure μ_*^Φ of the process Φ attracts all initial distributions (with finite first moments) of this process at a geometric rate with respect to the *Fortet-Mourier distance* (see, e.g., [Las95, p. 236] or [LY94, p. 46] for the definition of the Fortet-Mourier norm; cf. also [Dud66], where the equivalent *Dudley norm* is discussed). Such a metric, also known in the literature as a *bounded-Lipschitz distance*, is defined on the cone of non-negative finite Borel measures on X, and induces the topology of weak convergence of such measures ([Dud66, Theorems 8 and 9]). As a strightforward cosequence of Theorem 4.1, we can deduce a result that refers to stability of the chain $\{Y_n\}_{n\in\mathbb{N}_0}$ itself (see Corollary 4.2 in [H2]).

In the proof of the main theorem of [H2] (that is, Theorem 4.1) we take advantage of the asymptotic coupling techniques introduced by M. Hairer [Hai02] (cf. also [Ś11] and [D3]), and apply the results of R. Kapica and M. Ślęczka [KS20], which are based on them (to understand the main ideas behind [KS20, Theorem 2.1], we refer the reader to the Appendix in [H2], where we give an outline of its proof; see also Section 2 in [H4], with particular reference to Lemmas 2.2 and 2.3, where we prove an intermediate result, albeit in a slightly stronger version than that given in [KS20]).

Remarks on the assumptions imposed on the model under study.

Conditions imposed on the model components, that is, semiflows S_i, $i\in I$, a matrix $[\pi_{ij}]_{i,j\in I}$ of their (place-dependent) probabilities, continuous transformations w_θ, $\theta\in\Theta$, deciding about post-jump locations, and the associated set $\{p_\theta\}_{\theta\in\Theta}$ of (state-dependent) probability density functions (with respect to ϑ), are all listed in Section 2 of [H2] (see also Section 4.3.3 of this Summary; cf. Section 4 in [H4] and Section 5 in [H5]). **The reasonableness of these hypotheses are discussed in detail in Section 3 of [H2], where it is clarified that the semiflows enjoying them can be, e.g., generated by certain differential equations involving dissipative operators** (see also Section 4 in [E2] or Section 3 in [H3]; cf. [IK02, CK19]). Additional condition involving the jump intensity λ, as well as the constants appearing in the undertaken assumptions, is required to ensure the *Foster-Lyapunov drift condition*, appearing in the criterion of R. Kapica and M. Ślęczka ([KS20, Theorem 2.1]) on geometric ergodicity, to which we pertain in the proof of our main result. Conditions of this type are commonly used to study ergodic properties of Markov processes (cf. [MT93a, MT93b]). Moreover, a similar requirement appears, e.g., in [Las95, Proposition 5.1], where a Poisson driven stochastic differential equation is considered.

Another key result of [H2] is **establishing a one-to-one correspondence between the sets of invariant probability measures of the chain Φ and those of the PDMP Ψ** (Theorem 4.4 in [H2]). Namely, we show that whenever μ^Φ is an invariant probability measure of Φ, then $\mu^\Psi := \mu^\Phi G$ is an invariant probability measure of Ψ, and $\mu^\Psi W = \mu^\Phi$ (the converse statement is also true), where $G, W : X\times\mathcal{B}(X)\to[0,1]$ are the stochastic kernels defined as follows:

$$G((y,i),A)=\int_0^\infty \lambda e^{-\lambda t}\mathbb{1}_A\left(S_i(t,y),i\right)\,dt, \tag{6}$$

$$W((y,i),A)=\sum_{j\in I}\int_{\mathrm{supp}(\nu)}\int_\Theta \mathbb{1}_A\left(w_\theta(y)+v,j\right)\pi_{ij}\left(w_\theta(y)+v\right)p_\theta(y)\,\vartheta(d\theta)\,\nu(dv) \tag{7}$$

for each $(y,i)\in X$ and each $A\in\mathcal{B}(X)$. Let us indicate that, to prove the above, we additionally assume that the measure ϑ, given on the set Θ, is finite. The proof of Theorem 4.4 is based on techniques similar to those employed in [Hor08, Theorem 5.3.1] and [BLBMZ15, Propositions 2.1, 2.4] (in Lemmas 6.3 and 6.5 in [H2], we demonstrate that the transition semigroup $\{P(t)\}_{t\in\mathbb{R}_+}$ of Ψ is Feller and stochastically continuous, respectively). Moreover, in the proof of Theorem 4.4 in [H2] we also apply several results from the theory of semigroups of linear operators in Banach spaces, among others, those concerning weak infinitesimal operators of transition functions (see [Dyn65, pp. 36–43, 47–61] and [Dyn00, pp. 437–448]).

Combining Theorems 4.1 and 4.4 in [H2], we obtain that the PDMP Ψ has the unique invariant probability measure μ_*^Ψ (Corollary 4.5 in [H2]).

Then, **exponential ergodicity of the considered system with a general jump kernel J (but with probabilities π_{ij}, $i,j\in I$, which are constant, that is, independent of the current position of the system) is addressed in Propositions 7.1, 7.2 and Theorem 7.1 of** [E2]. To be more precise, the main goal of [E2] is to provide relatively easy to check conditions on the kernel J and the semiflows S_i, $i\in I$, which would guarantee that both the transition operator of the chain Φ and the transition semigroup of the process Ψ are exponentially ergodic in the Fortet-Mourier distance. Summary of [E2].

The general strategy of our approach, partially inspired by the techniques used in the proof of [CH15, Theorem 1.4], is as follows:

(I) showing that, whenever J enjoys some strengthened form of the Feller property, there exists a one-to-one correspondence between the set of invariant distributions of the process Ψ and those of the associated chain Φ (Theorem 5.1 in [E2]);

(II) noting that the existence of an appropriate coupling $(\Phi^{(1)},\Phi^{(2)})$ between two copies of Φ, such that the mean distance between them decreases geometrically with time, in conjunction with the so-called *Foster-Lyapunov drift condition* (cf. [DMS14, Definition 6.23]) and the Feller property imposed on P, ensures the exponential ergodicity of Φ (Lemma 6.1 in [E2]);

(III) proving that, for a given coupling $(\Phi^{(1)},\Phi^{(2)})$ of the chain Φ enjoying the property indicated in step (II), the corresponding coupling for the process Ψ has an analogous property, provided that the semiflows S_i fulfil a certain Lipschitz-type condition (Lemma 6.2 in [E2]);

(IV) observing that, under suitable assumptions on the semiflows S_i and the kernel J, providing all the requirements mentioned in steps (I)–(III), the existence of an appropriate coupling of Φ implies the exponential ergodicity of the process Ψ (Theorem 6.1 in [E2]);

(V) employing some additional hypotheses which, together with the previous ones, ensure that the coupling mentioned in (II) exists, which leads us directly to the main result of [E2], that is, Theorem 7.1 (we require the existence of a substochastic kernel Q_J on Y^2 with certain specific properties, in the spirit of [H4] or [KS20], such that

$$Q_J((y_1,y_2),\cdot\times Y)\le J(y_1,\cdot)\quad\text{and}\quad Q_J((y_1,y_2),Y\times\cdot)\le J(y_2,\cdot),$$

which further allows to construct a substochastic kernel $\widetilde{Q}_P$ on X^2, having the analogous properties with respect to P; see Lemma 7.1 in [E2]);

(VI) observing that the transition function of the desired coupling can be then defined as the sum of $\widetilde{Q}_P$ and a suitable complementary kernel (Proposition 7.1 n [E2]).

What is especially noteworthy here is the fact that this approach also elucidates the way in which the exponential ergodicity of the PDMP Ψ is inherited from the same property for the associated chain Φ (see steps (I) and (III)). We shall observe the same effect while disccusing, in the subsequent sections, various properties of invariant probability measures of Φ and Ψ, as well as limit theorems for them.

In the end, let us, however, highlight that it is still an open problem, whether the exponential ergodicity of the discrete-time model can imply the analogous property for the associated PDMP in the case of the model with place-dependant probability matrices $[\pi_{ij}(y)]_{i,j\in I}$, $y \in Y$.

The SLLN for the considered class of PDMPs and associated chains describing their post-jump locations

Upon establishing that a given process has the unique stationary distribution, it is natural to inquire about approximating it, e.g., through averages of (many) sample trajectories of this process. In this regard, in [H2], apart from proving exponential ergodicity, we also **demonstrate the SLLN for both the chain Φ** (Theorem 4.3 in [H2]) **and the PDMP Ψ** (Theorem 4.7 in [H2]). Summary of [H2].

The SLLN for Φ, stating that, for each bounded Lipschitz observable $g : X \to \mathbb{R}$, the averages $n^{-1}\sum_{k=0}^{n-1} g(\Phi_k)$ converge almost surely to $\int_X g\,d\mu_*^{\Phi}$ (with μ_*^{Φ} denoting the unique invariant probability measure for Φ), is derived from its geometric ergodicity (established in Theorem 4.1 in [H2]) and a modified version of [Shi03, Theorem 2.1], presented as Theorem 6.2 in [H2], which is a version of the SLLN for *mixing-type Markov chains*. While the original result by A. Shirikyan ([Shi03, Theorem 2.1]) is formulated for Markov chains evolving in Hilbert spaces, a careful analysis of its proof reveals that it can be easily adapted to apply to the case of Polish state spaces as well (Theorem 6.2 in [H2]).

Using a *martingale method* (cf. [BLBMZ15]), we then establish the SLLN for the PDMP Ψ based on the already proven SLLN for Φ and the one-to-one correspondence between invariant probability measures of Ψ and Φ (established in Theorem 4.4 in [H2]). To elaborate, our approach involves comparing the averages $t^{-1}\int_0^t g(\Psi(s))\,ds$ and $n^{-1}\sum_{k=0}^{N_t-1} Gg(\Phi_k)$, where G is defined by (6), $g : X \to \mathbb{R}$ represents some bounded Lipschitz observable and $\{N_t\}_{t\in\mathbb{R}_+}$ denotes the renewal counting process with arrival times τ_n, i.e. $N_t := \max\{n \in \mathbb{N}_0 : \tau_n \le t\}$ for $t \in \mathbb{R}_+$. We show, using arguments similar to those employed in the proof of [BLBMZ15, Lemma 2.5], that the difference between these two averages vanishes as $t \to \infty$. The subsequent steps then follow from Theorems 4.3 and 4.4 in [H2].

Properties of invariant probability measures of the considered class of PDMPs and associated chains describing their post-jump locations

Knowing that both the transition operator P of the chain Φ and the transition semigroup $\{P(t)\}_{t\in\mathbb{R}_+}$ of the process Ψ possess unique invariant probability measures, we then examine their properties.

In [H3], for any positive jump intensity λ, we investigate a version of the PDMP Ψ_λ (as well as the chain Φ_λ of its post-jump locations), given by (1) and (2), whose deterministic behavior between random jumps is governed by a single semiflow S, and whose jump mechanism is determined by a specific kernel J, given by Summary of [H3].

$$J(y,B)=\int_\Theta \mathbb{1}_B\left(w_\theta(y)\right)p_\theta(y)\vartheta(d\theta) \quad \text{for each} \quad y\in Y \quad \text{and} \quad B\in\mathcal{B}(Y). \tag{8}$$

The objective of [H3] is to **demonstrate the continuous (in the Fortet-Mourier distance) dependence of the invariant probability measures $\mu_*^{\Phi_\lambda}$ and $\mu_*^{\Psi_\lambda}$ (of Φ_λ and Ψ_λ, respectively) on the rate λ of a Poisson process, determining the frequency of jumps** (see Theorems 5.2 and 5.3 in [H3]; cf. Section 4.3.3 of this Summary). The ideas underlying the proofs of these results can be briefly summarized as follows. First of all, we observe that, for any $\lambda>0$ and any μ belonging to a certain subspace of the Banach space $\mathcal{M}_{\text{sig}}(Y)$ of all *finite signed Borel measures* on Y, one can express a measure μP_λ, with P_λ denoting a regular Markov operator induced by the transition law

$$\begin{aligned} P_\lambda(y,B)&=\int_0^\infty \lambda e^{-\lambda t}J(S(t,y),B)\,dt\\ &=\int_0^\infty \lambda e^{-\lambda t}\int_\Theta \mathbb{1}_B\left(w_\theta(S(t,y)\right)p_\theta(S(t,y))\,\vartheta(d\theta)\,dt, \quad y\in Y,\ B\in\mathcal{B}(Y), \end{aligned} \tag{9}$$

of the chain Φ_λ, as an appropriate *Bochner integral* (for definition see, e.g., [DU77] or Section 2 in [H3]), i.e.

$$\mu P_\lambda=\int_0^\infty \lambda e^{-\lambda t}\mu\Pi_{(t)}\,dt, \tag{10}$$

where

$$\Pi_{(t)}(y,B):=\int_\Theta \mathbb{1}_B\left(w_\theta(S(t,y))\right)p_\theta(S(t,y))\,\vartheta(d\theta) \quad \text{for} \quad y\in Y,\ B\in\mathcal{B}(Y)$$

(see Lemma 4.1 in [H3]). Then Lemma 4.2 in [H3] establishes that for each $\mu\in\mathcal{M}_{\text{sig}}(Y)$ the Fortet-Mourier norm of the measure $\mu\Pi_{(t)}$ is not greater than that of μ multiplied by a constant (dependent on t and the model parameters, excluding λ). In Lemma 4.4 in [H3], we, in turn, demonstrate that the map $(\lambda,\mu)\mapsto\mu P_\lambda$ is jointly continuous. Referring to Theorem 4.1 in [H2], we further get that (under its assumptions) the chain Φ is geometrically ergodic (in the Fortet-Mourier distance). Moreover, we prove that, for any λ from some interval $[\lambda_{\min},\lambda_{\max}]$ and any probability measure μ with finite first moment, the Fortet-Mourier distance between μP_λ^n and $\mu_*^{\Phi_\lambda}$ vanishes uniformly with respect to λ (Lemma 4.5 in [H3]). Finally, using all of the above, we can deduce the continuity of the map $\lambda\mapsto\mu_*^{\Phi_\lambda}$ in the topology of weak convergence of probability measures (Theorem 5.2 in [H3]). As a consequence, keeping in mind the one-to-one correspondence between $\mu_*^{\Phi_\lambda}$ and $\mu_*^{\Psi_\lambda}$, established in Theorem 4.4 in [H2], we get the continuity of $\lambda\mapsto\mu_*^{\Psi_\lambda}$ (Theorem 5.3 in [H3]).

While limit theorems (such as the SLLN or the CLT) provide the theoretical foundation for successful approximation of the invariant measures by means of observing or simulating (many) sample trajectories of the processes, this result asserts the stability of this procedure, at least locally in parameter space. It is a prerequisite for the development of a *bifurcation theory*. Moreover, even stronger regularity of this dependence on parameter (i.e., differentiability in a suitable norm in the space of measures) would be needed for applications in control theory or for parameter estimation (see, e.g., [GHLSG19]).

Let us also indicate that, since the considered mathematical model is intended to describe certain real-world phenomena (such as gene autoregulation, gene expression or cell division; cf. [H2] and also [HHS16, MTKY13, LM99]), **a continuous dependence of the invariant measure on the model parameter is desirable.**

Summary of [E3].

On the other hand, in [E3], we consider the versions of the processes Φ and Ψ with a state space $X := Y \times I$ such that Y is a closed (but not necessarily bounded, in contrast to [BLBMZ15]) subset of $\mathbb{R}^d$. The jump mechanism is described by a kernel J, determined by (4), with Θ being either an interval in $\mathbb{R}$ or a finite set, and $\nu = \delta_0$. The main goal of [E3] is to **provide certain verifiable conditions that would imply the absolute continuity of all the stationary distributions of the PDMP Ψ which correspond to ergodic stationary distributions of the associated chain Φ** (see Theorem 3.2 in [E3]). The absolute continuity is understood here to hold with respect to the product measure $\bar{l}_d$ of the d-dimensional Lebesgue measure and the counting measure on I. In fact, the problem reduces to examining the invariant distributions of the chain Φ of post-jump locations.

It should be emphasized that the hypotheses of Theorem 4.4 in [H2] do not ensure that the unique (and thus ergodic) stationary distribution of Φ (or that of Ψ) is absolutely continuous. The simplest example illustrating this claim is a system including only one transformation $w_1 \equiv 0$, for which the Dirac measure at 0 is the unique stationary distribution.

On the other hand, it is well known that, whenever the transition operator of a Markov chain preserves the absolute continuity of measures, then any ergodic stationary distribution of the chain must be either singular or absolutely continuous (see [LM94, Lemma 2.2 with Remark 2.1]; cf. [BH12, Theorem 6]). As clarified in Lemma 3.1 in [E3], this is the case for the chain Φ if, for instance, all the transformations w_θ and $S_i(t,\cdot)$ are non-singular with respect to the Lebesgue measure. Yet, as shown in Example 5.2 in [E3], even under this assumption, the conditions imposed in [H2] do not guarantee that the unique invariant distribution of the chain Φ and, thus, that of the PDMP Ψ, is absolutely continuous. It should be also stressed that, in general, the singularity of some of the transformations w_θ does not necessarily exclude the absolute continuity of invariant measures as well (see, e.g., [Löc18]).

Obviously, the above-mentioned absolute continuity/singularity dichotomy significantly simplifies the analysis, since, in such a setting, we only need to guarantee that the 'continuous part' of a given ergodic invariant distribution μ_*^Φ of Φ is non-trivial. One way to achieve this is to provide the existence of an open $\bar{l}_d$-small set (in the sense of [MT93b]) that is *uniformly accessible* from some measurable subset of X with positive measure μ_*^Φ in a specified number of steps (see Proposition 3.1 in [E3]).

We do this, following certain ideas of [BLBMZ15] (cf. Lemma 3.3 in [E3]). Furthermore, if the chain Φ is asymptotically stable (which is the case, e.g., under the hypotheses employed in [H2]), and (y_0, i_0) belongs to the support of μ_*^Φ, then the Portmanteau theorem ([Kle13, Theorem 2.1]) ensures that every open neighbourhood of (y_0, i_0) is uniformly accessible from some other (sufficiently small) neighbourhood of this point with positive measure μ_*^Φ in a given number of steps (cf. Corollary 3.1 in [E3]). In general, the latter may, however, be difficult to verify directly, and the argument works only if the chain is asymptotically stable. Therefore, we also propose a more practical condition ensuring the accessibility (cf. Lemma 3.4 in [E3]), which concerns the model components ($\{w_\theta\}_{\theta\in\Theta}$ and $\{S_i\}_{i\in I}$).

The CLT and the LIL for Markov processes that are exponentially ergodic in the Fortet-Mourier norm

To fully characterize the ergodic properties of the considered class of PDMPs and the associated chains describing their post-jump locations, it remains essential to establish conditions under which they satisfy the CLT and the LIL. We were initially optimistic, assuming that certain general versions of these limit theorems, proven for Markov processes which evolve in Polish metric spaces and which are exponentially ergodic in the Wasserstein distance (cf., e.g., [KW12, Theorem 2.1] and [BMS12, Theorem 1]), could be directly applied to our case (as it was while establishing the SLLN for Φ in [H2]). However, at that moment, **the existing versions of the CLT and the LIL were, to the best of our knowledge, not applicable to the random dynamical systems under study (we will elaborate on this issue later in this section). This served as a strong motivation to establish new, more practical versions of these limit theorems.**

The CLT is, beside the SLLN, the most fundamental limit theorem for random processes. For $\psi := \{\psi(t)\}_{t\in\mathbb{R}_+}$ ($\phi := \{\phi_n\}_{n\in\mathbb{N}_0}$) denoting a time-homogeneous Markov process (chain) evolving in a Polish metric space E with an arbitrary transition semigroup $\{P(t)\}_{t\in\mathbb{R}_+}$ (one-step transition probability function P) and the unique invariant probability distribution μ_* (either for ψ or for ϕ, depending on the context), $g : E \to \mathbb{R}$ denoting a bounded Lipschitz (or, at least, Borel measurable and square integrable with respect to μ_*) observable and $\bar{g} := g - \int_E g\,d\mu_*$, we say that the process $\{\bar{g}(\psi(t))\}_{t\in\mathbb{R}_+}$ ($\{\bar{g}(\phi_n)\}_{n\in\mathbb{N}_0}$) obeys the CLT if the average

$$\frac{1}{\sqrt{t}}\int_0^t \bar{g}\left(\psi(s)\right)ds \qquad \left(\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\bar{g}\left(\phi_i\right)\right)$$

converges in law, as $t \to \infty$ ($n \to \infty$), to a centered normal random variable.

The LIL can be, in turn, viewed as a refinement of the SLLN. It improves the convergence rate in the SLLN from $\mathcal{O}(t)$ to $\mathcal{O}(\ln(\ln(t)))$. More specifically, it provides the precise values of the lower and upper limit of almost all sequences formed by the properly scaled integrals (partial sums) of the sample paths of the stochastic process under study. Moreover, the LIL gives an interesting illustration of the difference between almost sure and distributional statements, such as the CLT. Using the notation already introduced above, we say that the process $\{\bar{g}(\psi(t))\}_{t\in\mathbb{R}_+}$ ($\{\bar{g}(\phi_n)\}_{n\in\mathbb{N}_0}$) obeys the LIL if

$$\limsup_{t\to\infty}\frac{\int_0^t \bar{g}(\psi(s))ds}{\sqrt{2t\ln(\ln(t))}} = \bar{\sigma}(\bar{g}) \quad \text{and} \quad \liminf_{t\to\infty}\frac{\int_0^t \bar{g}(\psi(s))ds}{\sqrt{2t\ln(\ln(t))}} = -\bar{\sigma}(\bar{g}) \quad \text{a.s.}$$

$$\left(\limsup_{n\to\infty}\frac{\sum_{i=0}^{n-1}\bar{g}\left(\phi_i\right)}{\sqrt{2n\ln(\ln(n))}} = \sigma(\bar{g}) \quad \text{and} \quad \liminf_{n\to\infty}\frac{\sum_{i=0}^{n-1}\bar{g}(\phi_i)}{\sqrt{2n\ln(\ln(n))}} = -\sigma(\bar{g}) \quad \text{a.s.}\right)$$

for some $0 < \bar{\sigma}(\bar{g}) < \infty$ ($0 < \sigma(\bar{g}) < \infty$).

Historical background of research on the limit theorems for Markov processes.

Initially formulated for independent and identically distributed random variables, the CLT and the LIL were thereafter generalized to martingales (see, e.g., [Lév35] for the CLT and [HS73, HH80] for the LIL), which has constituted a background for proving various versions of these limit theorems pertaining to Markov processes.

Let us first discuss the results for (discrete-time) Markov chains. The first results in this field deal with stationary Markov chains for which the existence of a *μ_*-square integrable solution to the Poisson equation* is guaranteed (cf. the versions of the CLT in [GL78, GL81, DL01]; numerous results related to the classical LIL are summarized in [Bin86]). During the later years, many attempts have been made to relax this assumption. For instance, [KV86] refers to the so-called *reversible Markov chains* and is based on approximating (in a certain sense) the solutions of the Poisson equation (see also [Lim00] for a version of the LIL for reversible Markov processes), while [MW00] and [ZW08] introduce appropriate testable conditions, for the CLT and the LIL, respectively, relying on the convergence of some series. Another noteworthy article is [JT20], where the principal hypothesis of [MW00] is reached by assuming a subgeometric rate of convergence (in terms of the Wasserstein distance) of Markov chain's distribution to its stationary one.

In recent times, however, most attention has been paid to *non-stationary Markov chains* (that is, those whose initial distribution is not necessarily the stationary one). Some classical results on the CLT and the LIL in this case can be found in [MT93b]. They involve *positive Harris reccurent and aperiodic Markov chains* (or, equivalently, those which are *irreducible* and ergodic in the total variation norm), for which a drift condition towards petite sets is fulfilled (which guarantees the existence of a suitable solution to the Poisson equation). Such requirements are, however, practically unattainable in non-locally compact state spaces. In this connection, in order to demonstrate limit theorems for geometrically ergodic (in the Wasserstein distance) Markov chains evolving in Polish spaces, new approaches have been proposed in a few recent papers, including [BMS12] and [GHSZ19]. Importantly, a solution to the Poisson equation is not required there. Our results in [H4] and [H5] are of a similar nature, although – from the point of view of some applications (cf. the relevant examples discussed in Section 4 of [H4] and Section 5 of [H5]) – they are more practical.

Summary of [H4] and [H5].

In [H4] and [H5] we establish certain versions of the CLT and the LIL (in the case of the LIL, we even prove its functional variant, the so-called *Strassen invariance principle*), respectively, for a subclass of non-stationary Markov chains evolving in general (Polish) metric spaces, based on a kind of geometric mixing in the Fortet-Mourier distance (see, e.g., [Hai02] for the precise formulation) **and the Foster-Lyapunov type condition ('second order' for the CLT and 'higher than second order' for the LIL).** As mentioned above, similar results, although based on geometric mixing in the Wasserstein distance (see, e.g., [KW12, p. 5] for its definition), are stated in [GHSZ19] and [BMS12]. Additionally, the mixing condition assumed in [GHSZ19] and [BMS12] is of a different nature than that in our papers, where **no dependance on the distance between initial measures is required**. More precisely, in [H4] and [H5] the proofs of the main results (that is, Theorem 3.2 in [H4] and Theorem 4.7 in [H5]) are based on the condition of the form: for any two Borel probability measures μ_1 and μ_2, there exist a *Lyapunov function* $V: E \to \mathbb{R}_+$ (that is, a map which is continuous, bounded on bounded sets, and, in the case of unbounded E, satisfies $\lim_{\rho(x,\bar{x})\to\infty} V(x) = \infty$ for some fixed point $\bar{x} \in E$) and some constants $c > 0$ and $q \in (0,1)$ for which

$$d_{\mathrm{FM}}\left(\mu_1 P^n, \mu_2 P^n\right) \le c q^n \left(1 + \int_E V\, d\left(\mu_1 + \mu_2\right)\right) \quad \text{for any} \quad n \in \mathbb{N}_0, \tag{11}$$

where d_{FM} stands for the Fortet-Mourier metric (which pertains to the ideas employed in [Hai02]). In [BMS12] and [GHSZ19], in turn, it is required that for any two Borel probability measures μ_1 and μ_2 (with finite first moments), there exist constants $c > 0$ and $q \in (0,1)$ such that

$$d_{\mathrm{W}}\left(\mu_1 P^n, \mu_2 P^n\right) \le c q^n d_{\mathrm{W}}\left(\mu_1, \mu_2\right) \quad \text{for any} \quad n \in \mathbb{N}_0, \tag{12}$$

where d_{W} stands for the Wassertein metric.

The motivation to replace condition (12) with condition (11) derives from our research on certain random dynamical systems, discussed earlier in this Summary (see Section 4.3.2: *PDMPs driven by randomly switched semiflows*, in particular definitions (1)–(5)) **and applied mainly in molecular biology** (cf. [MTKY13, HHS16, LM99] and [H2]), **to which we have not been able to apply neither [BMS12, Theorem 1] nor [GHSZ19, Theorem 5.1] directly. This is primarily caused by the fact that, upon certain general conditions imposed on the model (which appear to be reasonable in most applications), inequality (12) seems to be difficult or even impossible to achieve, whilst the same conditions naturally imply (11), as shown, e.g., in Theorem 4.1 in [H2].**

It is also worth stressing that we do not assume condition (11) directly, as it is usually not straightforward to derive from its definition. Instead, we propose a set of conditions, relatively easy to verify, which yield not only the geometric mixing property (11), but also the existence of the unique invariant probability measure (due to [KS20, Theorem 2.1]), **as well as the desired assertions, that is, the CLT (Theorem 3.2 in [H4]) and the LIL (Theorem 4.7 in [H5])**.

The class of non-stationary Markov-Feller chains for which we establish the CLT and the LIL may be characterised by just two properties. The first one concerns the existence of an appropriate Markovian coupling whose transition function can be decomposed into two parts, one of which is contractive and dominant in some sense. The construction of such a coupling is adapted from [Cza18, KS20], which, in turn, is inspired by the prominent results of M. Hairer [Hai02]. Within this framework, we provide a geometric estimate of the mean distance between two coupled copies of the examined chain. This result, stated in Lemmas 2.2 and 2.3 in [H4], slightly generalizes the geometric mixing property obtained by R. Kapica and M. Ślęczka while proving [KS20, Theorem 2.1]. The precise proofs of these lemmas are interesting themselves, as well as they also clarify the reasoning presented in [KS20]. In fact, Lemmas 2.2 and 2.3 play a key role in both [H4] and [H5]. The second property which characterises the distiguished class of Markov-Feller chains is enjoying by their transition operators a non-linear Lyapunov-type condition. One of the simplest classes of Markov chains achieving the desired properties are those arised from random IFSs with an arbitrary number of transformations, which are assumed to be contractive on average, such as those considered in [Wer05, HS06, Ś11, KS20].

The proof of Theorem 3.2 in [H4] also appeals to the results of M. Maxwell and M. Woodroof [MW00], which make it more concise and less technical than the classical proofs, based directly on martingale methods. The proofs in [D2] and [Hor16] are carried out in the same spirit, although only for some specific cases. It is also worth mentioning here that **the conditions proposed in [H4] yield *the Donsker invariance principle for the CLT* (cf. [Bil99]), provided that the Markov chain is stationary**.

On the other hand, some proof techniques, employed in [H5], are adapted from the articles [BMS12] and [D1], which both pertain to the martingale results by C.C. Heyde and D.J. Scott [HS73]. At the beginning of Section 4 in [H5], we also present a few general observations concerning martingales (see Lemmas 3.2–3.5), which are then useful in the proof of the main result of that paper, that is, Theorem 4.7.

Finally, **to justify the usefulness of the newly established general versions of the CLT (Theorem 3.2 in [H4]) and the LIL (Theorem 4.7 in [H5]), we apply them to prove the CLT (Theorem 4.1 in [H4]) and the LIL (Theorem 5.2 in [H5]) for a specific non-stationary Markov chain Φ, whose transition probability function is given by** (5), and whose possible applications are discussed in Section 4.3.2: *PDMPs driven by randomly switched semiflows* of this Summary.

Let us now turn to limit theorems for continuous-time Markov processes. Similarly to the discrete case, these theorems were initially established for stationary processes (e.g., for ergodic processes with *normal generators* in [Hol05], extending the results of [GL81]; see also [OLK12]). Subsequently, a few results for non-stationary Markov processes with continuous-time parameters have emerged. Probably the most general result of this kind to date is stated as [KW12, Theorem 2.1] (its analog in the discrete-time case is given as [GHSZ19, Theorem 5.1]).

Summary of [E1].

In [E1] we establish a version of the CLT for Markov-Feller continuous time processes (with a Polish state space) that are exponentially ergodic in the Fortet-Mourier distance and enjoy a continuous form of the Foster-Lyapunov condition. This result is, again, mainly inspired by the inability to directly apply the existing version of the CLT ([KW12, Theorem 2.1] by T. Komorowski and A. Walczuk in this case) to some subclass of PDMPs, at least under certain (relatively natural) conditions imposed in Proposition 7.2 in [E2] (cf. Section 4.3.2: *PDMPs driven by randomly switched semiflows* of this Summary).

The problem lies in accomplishing the exponential mixing in the sense of condition (H1), employed in [KW12], which requires a form of the Lipschitz continuity of each $P(t)$ with respect to the *Wasserstein distance* d_{W}. More precisely, the authors assume the existence of $\gamma > 0$ and $\bar{c} < \infty$ such that for any two Borel probability measures μ_1 and μ_2 (with finite first moments) the following holds:

$$d_{\mathrm{W}}\left(\mu_1 P(t), \mu_2 P(t)\right) \leq \bar{c} e^{-\gamma t} d_{\mathrm{W}}\left(\mu_1, \mu_2\right) \quad \text{for any} \quad t \in \mathbb{R}_+. \tag{13}$$

We therefore recognize the need to introduce a new, more useful criterion that would involve a weaker form of the above requirement (as in the discrete case). In this connection, instead of (13) we assume in [E1] that there is some $\gamma > 0$ such that, for any two Borel probability measures μ_1 and μ_2, there exist a continuous function $V : E \to [0, \infty)$ and constants $\bar{c} > 0$, $\delta \in (0, 1)$, for which

$$d_{\mathrm{FM}}\left(\mu P(t), \nu P(t)\right) \leq \bar{c} e^{-\gamma t} \left(\int_E V \, d(\mu + \nu) + 1 \right)^{\delta} \quad \text{for any} \quad t \in \mathbb{R}_+. \tag{14}$$

Besides, an additional advantage of our approach is that this metric is weaker than the Wasserstein one, among others, in a manner enabling the use of a coupling argument (introduced by M. Hairer in [Hai02]) to reach an exponential mixing property w.r.t. d_{FM}, which fails while demanding it in terms of d_{W}.

As mentioned above, the proof of our main result, that is, Theorem 2.1 in [E1], is in many places based on the reasoning presented in [KW12]. Nevertheless, it should be emphasized that without a Lipschitz type assumption on the semigroup $\{P(t)\}_{t \in \mathbb{R}_+}$, such as (13) (or its discrete-time analogue, employed, e.g., in [GHSZ19]; cf. also [BMS12]), proving the principal limit theorems, like the CLT or the LIL, requires some more subtle arguments, which is also reflected in [H4], [H5] or [KPS13]. Most importantly, under condition (14), the so-called *corrector function* $\chi : E \to \mathbb{R}$, given by

$$\chi(x) = \int_0^\infty P(t) \bar{g}(x) \, dt \quad \text{for any} \quad x \in E,$$

does not need to be Lipschitzian (which is a meaningful argument in the proof of [KW12, Theorem 2.1]), but is only continuous. Another problem arising in our setting is that the weak convergence of the process distribution (towards the stationary one), guaranteed by (14), yields the convergence of the corresponding integrals as long as the integrands are (apart of being continuous) bounded, which is not required while using the Wasserstein distance. We have overcome this obstacle (see Lemma 3.9 in [E1]) by making the use of [Bog07, Lemma 8.4.3], which allows replacing the boundedness of the integrand by its uniform integrability with respect to the family of measures constituing the convergent sequence under consideration.

In Section 4 of [E1], we also elaborate on a representation of σ^2, that is, the variance of the limit normal distribution involved in Theorem 2.1, while in Section 5 of [E1] we justify the usefulness of the main result (Theorem 2.1) by applying it to establish the CLT for the PDMP Ψ considered in [E2] (see Corollary 5.1 in [E1]).

In the future, our goal is to establish an equivalent of Theorem 2.1 in [E1], which would address the LIL for Markov-Feller continuous-time processes (with a Polish state space) that are exponentially ergodic in the Fortet-Mourier distance and satisfy a continuous form of the Foster-Lyapunov condition. Some inspiration for this project can be found not only in [E1], but also in [KPS13].

At the end of this section, let us shortly summarize the results presented in [E4]. **The main goal of that paper is to prove the validity of the LIL for the PDMP Ψ introduced in** [H2] (and discussed in detail in Section 4.3.2: *PDMPs driven by randomly switched semiflows* of this Summary). **Our method of proof is intentionally such that the result for** Ψ, defined as in (1) and (2), **is derived from the validity of the LIL for the associated chain Φ given by its post-jump locations**, already etablished in Theorem 5.2 in [H5]. Summary of [E4].

Essentially, our method of proof splits the problem into two subproblems that are analyzed separately (in Sections 3.2 and 3.1 of [E4], respectively). One subproblem can be addressed using a version of the LIL for certain square integrable martingales, whose proof draws heavily on [HS73, Theorem 1] and asymptotic coupling methods, used in a similar way as in the proof of Lemma 2.2 in [H4]. Another builds on the validity of the LIL for a Markov chain Φ associated to PDMP Ψ.

Summarizing this and the above sections, let us point out that the difficulty of our research arises from the fact that we do not impose any significant restrictions on the phase space of the considered process, such as compactness. The assumption of compactness (or even local compactness) would limit the applicability of the obtained results to finite-dimensional spaces only. However, we are aware that modeling real-world phenomena often requires functional spaces, which are infinite-dimensional (e.g., in the gene autoregulation model introduced in [HHS16], the phase space consists of continuous functions describing the concentration of chemical compounds at different points in the cell cytoplasm).

Random Interval Homeomorphisms

In [H6] **we are concerned with an IFS generated by orientation-preserving homeomorphisms on the interval $[0,1]$, previously examined (among others) by K. Czudek and T. Szarek in [CS20], and we prove that, besides the CLT (see [CS20, Theorem 4], it also obeys the LIL.** Summary of [H6].

Let us begin with recalling the definition of an admissible IFS. Let $f_1, \ldots, f_N$ be increasing homeomorphisms of the interval $[0,1]$ such that for every $x \in (0,1)$ there exist indexes $i, j \in \{1, \ldots, N\}$ for which $f_i(x) < x < f_j(x)$. It is assumed that all the homeomorphisms are differentiable at 0 and 1 with non-zero derivatives. Moreover, let $(p_1, ..., p_N)$ be a probability vector such that

$$\sum_{i=1}^{N} p_i \log\left(f_i'(0)\right) > 0 \quad \text{and} \quad \sum_{i=1}^{N} p_i \log\left(f_i'(1)\right) > 0.$$

The family $(f_1, \ldots, f_N; p_1, \ldots, p_N)$ is then called an ***admissible IFS***.

Now, let $(f_1, \ldots, f_N; p_1, \ldots, p_N)$ be an admissible IFS, and note that it generates a stochastic kernel $P : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$, which is given by

$$P(x, A) = \sum_{i=1}^{N} p_i \delta_x \left(f_i^{-1}(A)\right) \quad \text{for any} \quad x \in [0,1], \;\; A \in \mathcal{B}([0,1]). \tag{15}$$

By the continuity of the functions f_i, $i \in \{1, \ldots, N\}$, the corresponding regular Markov operator P is obviously Feller.

In what follows, let us denote by $\{X_n\}_{n\in\mathbb{N}}$ the Markov chain with transition probability function P, defined in the coordinate space $([0,1]^{\mathbb{N}}, \mathcal{B}([0,1]^{\mathbb{N}}))$, equipped with an appropriate collection $\{\mathbb{P}_\nu\}_\nu$ of probability measures on $\mathcal{B}([0,1]^{\mathbb{N}})$ such that $\mathbb{P}_\nu(X_{n+1} \in A | X_n = x) = P(x, A)$ and $\mathbb{P}_\nu(X_1 \in A) = \nu(A)$ for any probability measure ν on $[0,1]$.

The article [CS20], besides the proof of the CLT, also contains a simple proof of the unique ergodicity in the open interval $(0,1)$ for the Markov operator P, determined by (15). More precisely, [CS20, Theorem 1] says that P has the unique invariant probability measure μ_* on $(0,1)$ which is atomless, and [CS20, Theorem 2] yields the asymptotic stability of P in the space of probability measures supported on $(0,1)$ (i.e., $\mu_*((0,1)) = 1$ and $\lim_{n\to\infty} \int_{[0,1]} \varphi \, d(\mu P^n) = \int_{[0,1]} \varphi \, d\mu_*$ for every continuous function $\varphi : [0,1] \to \mathbb{R}$).

It shall be highlighted that, historically, such a phenomenon was first proved by L. Alsedá and M. Misiurewicz for some function systems consisting of piecewise linear homeomorphisms (see [AM14]). More general IFSs were then considered by M. Gharaei and A. J. Homburg in [GH17] ([CS20, Theorems 1 and 2] are just the repetition of these arguments). Recently, D. Malicet has also obtained the unique ergodicity as a consequence of the contraction principle for time homogeneous random walks on the topological group of homeomorphisms defined on the circle (see [Mal17]). His proof, in turn, is based upon the invariance principle of A. Avila and M. Viana (see [AV10]).

Let $\varphi : [0,1] \to \mathbb{R}$ be an arbitrary Lipschitz function satisfying $\int_{[0,1]} \varphi \, d\mu_* = 0$, where μ_* denotes the unique invariant probability measure of P on $(0,1)$, and let $\{X_n^*\}_{n\in\mathbb{N}}$ stand for the stationary Markov chain with transition probability function P given by (15) (that is, the Markov chain with transition probability function P starting from its stationary distribution μ_*).

The proof of the CLT for $\{\varphi(X_n)\}_{n\in\mathbb{N}}$, which is given in [CS20], is based on the Maxwell–Woodroofe approach [MW00] for ergodic stationary Markov chains. The direct application of the results from [MW00] allows to prove the CLT for the stationary Markov chain $\{\varphi(X_n^*)\}_{n\in\mathbb{N}}$ (the so-called *annealed CLT*). On the other hand, if additionally some coupling techniques are applied to evaluate the distance between the Fourier transforms of the stationary and an arbitrary non-stationary Markov chain, then the so-called *quenched CLT* follows.

Similarly, in [H6], we also first establish the LIL for the stationary Markov chain $\{\varphi(X_n^*)\}_{n\in\mathbb{N}}$ (see Proposition in [H6]). The arguments in this case are based on the results by O. Zhao and M. Woodoofe [ZW08]. In our setting, the Zhao-Woodroofe criterion for the LIL for the chain $\{\varphi(X_n^*)\}_{n\in\mathbb{N}}$ takes the following form:

$$\sum_{n=1}^{\infty}\left(\frac{\ln(n)}{n}\right)^{3/2}\left\|\sum_{k=1}^{n}P^k\varphi\right\|_{L^2(\mu_*)}<\infty,$$

where $\|\cdot\|_{L^2(\mu_*)}$ denotes the L^2-norm with respect to the invariant probability measure μ_* of P on $(0,1)$ (cf. [ZW08, Theorem 1 and Corollary 1]). **Then, using some calculations given in [CS20], we prove the validity of the quenched LIL (see Theorem in [H6]).**

Quenched limit theorems (either CLTs or LILs) have been recently proved for various non-stationary Markov processes (see our papers [H4] and [H5], discussed in the prevous section, as well as the articles [DL01, LS05, KW12, OLK12, BMS12, GHSZ19], just to name a few). Most of them are formulated for Markov processes with transition probabilities satisfying the property of the spectral gap in the total variation or, at least, the Wasserstein/Fortet-Mourier norm. However, it is not known whether the Markow chain $\{X_n\}_{n\in\mathbb{N}}$ corresponding to an admissible IFS $(f_1,\ldots,f_N;p_1,\ldots,p_N)$ is such. Moreover, in the case when each of the functions f_i, $i\in\{1,\ldots,N\}$, has a fixed point at 0 and at 1, no cotracting condition (either that demanded in [H4], [H5] or that required in [BMS12, GHSZ19], both leading to geometric ergodicity of P to equilibrium) may hold. Therefore, to the best of our knowledge, none of the existing criteria for the quenched CLT or LIL could be applied directly.

In the future, it may be interesting to investigate the validation of the large deviations principle under Maxwell-Woodroofe or Zhao-Woodroofe-type conditions. As far as we know, there is no literature on this topic so far, and thus new methods have to be developed.

4.3.3 Presentation of the main theorems of the scientific achievement

[H1] *The e-property of asymptotically stable Markov-Feller operators*

Let (E, ρ) be a Polish metric space. The two main results of [H1] read as follows:

Theorem 4.1 (Theorem 3.1 in [H1]). *Let P be an asymptotically stable Markov-Feller operator with values in E. The set of points where P fails the e-property in $\mathrm{C}_b(E)$ is a set of first category, while the set of points at which P has the e-property in $\mathrm{C}_b(E)$ is dense.*

Theorem 4.2 (Theorem 3.5 in [H1]). *Let P be an asymptotically stable Markov-Feller operator with values in E, and let μ_* denote its invariant probability measure. Then P has the e-property in $\mathrm{C}_b(E)$ if there exists at least one point $z \in supp(\mu_*)$ at which P has the e-property in $\mathrm{C}_b(E)$.*

The combination of Theorems 4.1 and 4.2 leads to the following result by S.C. Hille, T. Szarek, and M. Ziemlańska [HSZ17]:

Theorem 4.3. *[HSZ17, Theorem 2.3] Let P be an asymptotically stable Markov-Feller operator with values in E, and let μ_* denote its invariant probability measure. If $\mathtt{Int}(\mathrm{supp}(\mu_*)) \neq \emptyset$, then P satisfies the e-property in $\mathrm{C}_b(E)$.*

Indeed, it is clear, according to Theorem 4.1, that if the interior of the support of an invariant probability measure of a Markov–Feller operator P is non-empty, then there exists at least one point in this support at which P has the e-property. This, in turn, implies, due to Theorem 4.2, that P has the e-property at any point.

Now, let $\mathrm{Lip}_{b,1}(E)$ be the subspace of the space $\mathrm{Lip}_b(E)$ given by

$$\mathrm{Lip}_{b,1}(E) := \{ f \in \mathrm{Lip}_b(E) : \ \|f\|_{\mathrm{BL}} \leq 1 \}, \tag{16}$$

where the norm $\|\cdot\|_{\mathrm{BL}}$ is defined as

$$\|f\|_{\mathrm{BL}} := \max\left\{ \|f\|_\infty, \ \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)} \right\} \quad \text{for any} \quad f \in \mathrm{Lip}_b(E). \tag{17}$$

The proof of Theorem 4.2 is based on the application of [HSZ17, Lemma 2.4], while, to prove Theorem 4.1, we, among others, use the following lemma:

Lemma 4.1 (Lemma 3.4 in [H1]). *A regular E-valued Markov operator P which is asymptotically stable and which has the e-property in $\mathrm{Lip}_{b,1}(E)$ at $z \in E$ also has the e-property in $\mathrm{C}_b(E)$ at $z \in E$.*

At this point, let us also emphasize that, thanks to Lemma 4.1, we can immediately extend the statement of [HSZ17, Theorem 2.3] from its original formulation, which is limited to the e-property in $\mathrm{Lip}_b(E)$, to the more general context of the e-property in $\mathrm{C}_b(E)$ (as in Theorem 4.3).

Finally, let us indicate that Section 4 of [H1] includes two important examples. In Example 1 we construct an asymptotically stable Markov-Feller operator P such that the set of points at which P fails the e-property in $\mathrm{C}_b(S)$ is dense, making it a non-trivial set of first category. The main aim of presenting this example is to demonstrate that Theorem 4.1 is a tight result. In Example 2 we construct an asymptotically stable Markov-Feller operator P such that the set of points where P fails the e-property in $\mathrm{C}_b(S)$ has positive Lebesgue measure, and hence it is uncountable.

The results presented in [H1] complement and specify the description of the relationship between the e-property and the asymptotic stability of Markov operators in general (Polish) metric spaces. Notably, they generalize [HSZ17, Theorem 2.3].

[H2] *Ergodic properties of some PDMP with application to gene expression modelling*

For an arbitrary Polish metric space (E,ρ), let us write $\mathcal{M}(E)$ and $\mathcal{M}_1(E)$ for the space of all finite non-negative Borel measures on E and its subset consisting of all probability measures, respectively. Moreover, for any given Borel measurable function $V : E \to \mathbb{R}_+$ and any $r > 0$, let us consider the subset $\mathcal{M}_{1,r}^V(E)$ of $\mathcal{M}_1(E)$ consisting of all measures with finite r-th moment with respect to V, that is,

$$\mathcal{M}_{1,r}^V(E) := \left\{\mu \in \mathcal{M}_1(E) : \int_E V^r \, d\mu < \infty\right\}.$$

To evaluate the distance between measures, we here use the Fortet-Mourier metric, which on $\mathcal{M}(E)$ is obviously given by

$$d_{\mathrm{FM}}(\mu,\nu) := \sup_{f\in \mathrm{Lip}_{b,1}(E)} \left|\int_E f \, d(\mu-\nu)\right| \quad \text{for any} \quad \mu,\nu \in \mathcal{M}(E), \tag{18}$$

where $\mathrm{Lip}_{b,1}(E)$ is defined in (16).

Consider a separable Banach space $(H, \|\cdot\|)$ and a closed subset Y of H (note that we can think of Y as a Polish metric space with the metric induced by the norm $\|\cdot\|$ in H). Let us also fix a topological measure space $(\Theta, \mathcal{B}(\Theta), \vartheta)$ with a finite Borel measure ϑ, and let I be a finite set equipped with the metric d given by $d(i,j) = 0$ for $i = j$ and $d(i,j) = 1$ otherwise.

We here investigate a PDMP $\Psi = \{(Y(t), \xi(t))\}_{t\in\mathbb{R}_+}$, evolving through jumps, occuring at random moments τ_n, $n \in \mathbb{N}$, which coincide with the jump times of a Poisson process with a given intensity $\lambda > 0$. The deterministic behaviour of the system between the jumps is governed by a collection $\{S_i\}_{i\in I}$ of semiflows, where $S_i : \mathbb{R}_+ \times Y \to Y$, and any state right after the jump is attained by a randomly selected continuous transformation $w_\theta : Y \to Y$, $\theta \in \Theta$ (a more detailed description can be found in Section 2 in [H2], Section 4 in [H4], Section 5 in [H5] or Section 4.3.2: *PDMPs driven by randomly switched semiflows* of this Summary).

Let $X := Y \times I$ be endowed with the metric $\rho_{\tilde{c}}$ given by

$$\rho_{\tilde{c}}\left((y_1,i_1),(y_2,i_2)\right) = \|y_1 - y_2\| + \tilde{c}\, d(i_1,i_2) \quad \text{for} \quad (y_1,i_1),(y_2,i_2) \in X,$$

where $\tilde{c} \geq 1$ is some sufficiently large constant (defined explicitly in [H2]).

Formally, we consider:

- an X-valued time-homogeneous Markov chain $\Phi := (Y_n, \xi_n)_{n\in\mathbb{N}_0}$ (cf. (2) or Eqs. (2.3) and (2.4) in [H2]) with transition law $P : X \times \mathcal{B}(X) \to [0,1]$ given by (5), where $\{\xi_n\}_{n\in\mathbb{N}_0}$ is an I-valued sequence of random variables describing the indexes of the 'currently active' semiflows,

- and an X-valued time-homogenous Markov process $\Psi := \{(Y(t), \xi(t))\}_{t\in\mathbb{R}_+}$ defined via interpolation of Φ by (1) (the transition semigroup of Ψ will be denoted by $\{P(t)\}_{t\in\mathbb{R}_+}$).

Let us detail the conditions that we impose on the model components, that is, semiflows S_i, $i \in I$, a matrix $[\pi_{ij}]_{i,j\in I}$ of their (place-dependent) probabilities, continuous transformations w_θ, $\theta \in \Theta$, deciding about post-jump locations, and the associated set $\{p_\theta\}_{\theta\in\Theta}$ of (state-dependent) probability density functions (with respect to ϑ). We assume that there exist $\bar{y} \in Y$, a function $\mathcal{L} : Y \to \mathbb{R}_+$ that is bounded on bounded sets, and constants $\alpha \in \mathbb{R}$, L, L_w, L_π, L_p, c_π, $c_p > 0$ such that

$$LL_w + \alpha/\lambda < 1, \tag{19}$$

and, for any $i, i_1, i_2 \in I$, $y_1, y_2 \in Y$, $t \in \mathbb{R}_+$, the following conditions hold:

$$\sup_{y\in Y} \int_0^\infty e^{-\lambda t} \int_\Theta \|w_\theta(S_i(t,\bar{y})) - \bar{y}\| \, p_\theta\left(S_i(t,y)\right) \vartheta(d\theta)\, dt < \infty, \tag{M1}$$

$$\|S_{i_1}(t,y_1) - S_{i_2}(t,y_2)\| \le Le^{\alpha t}\|y_1 - y_2\| + t\mathcal{L}(y_2)\, d(i_1,i_2), \tag{M2}$$

$$\int_\Theta \|w_\theta(y_1) - w_\theta(y_2)\| \, p_\theta\left(y_1\right) d\theta \le L_w \|y_1 - y_2\|, \tag{M3}$$

$$\sum_{j\in I} |\pi_{ij}(y_1) - \pi_{ij}(y_2)| \le L_\pi \|y_1 - y_2\|, \quad \int_\Theta |p_\theta\left(y_1\right) - p_\theta\left(y_2\right)| \, d\theta \le L_p \|y_1 - y_2\|, \tag{M4}$$

$$\sum_{j\in I} \pi_{i_1 j}(y_1) \wedge \pi_{i_2 j}(y_2) \ge c_\pi, \quad \int_{\Theta(y_1,y_2)} p_\theta\left(y_1\right) \wedge p_\theta\left(y_2\right) d\theta \ge c_p, \tag{M5}$$

where $\Theta(y_1, y_2) := \{\theta \in \Theta : \|w_\theta(y_1) - w_\theta(y_2)\| \le L_w\|y_1 - y_2\|\}$.

Interested readers are referred to Section 3 of [H2], where we discuss the reasonableness of the above-listed assumptions. Specifically, we provide justification for the attainability of condition (M2) within a broad class of semiflows acting on reflexive Banach spaces (particularly, Hilbert spaces). Furthermore, we elaborate on how such semiflows can be generated by specific differential equations involving dissipative operators (cf. [IK02, CK19]). We also observe that, in many cases, condition (M1) can be easily derived from the conjunction of (M2) and (M3).

Moreover, we observe that conditions (M3)–(M5), which ensure that the system of transformations $\{w_\theta\}_{\theta\in\Theta}$ is contractive on average, and which impose additional restrictions on probabilities and densities in the model, are also quite natural. Such properties are commonly demanded in the analysis of asymptotic properties of classical random IFSs (cf. [LY94] or [Sza03]), which are covered by our discrete-time model. The example discussed in [Ste01] highlights that assumptions formulated in a manner similar to (M4) cannot be omitted, even in the simplest scenarios. To elaborate, the system $\{(S_1, p), (S_2, 1-p)\}$, consisting of two contractions S_1 and S_2 and a positive continuous probability function p, may admit more than one invariant probability measure (unless, at least, the Dini continuity of p is assumed).

The main aim of the article [H2] is to provide a criterion on geometric ergodicity (in d_{FM}) of the chain Φ.

In what follows, let $V : X \to [0,\infty)$ be given by $V(y,i) = \|y - \bar{y}\|$ for any $(y,i) \in X$.

Theorem 4.4 (Theorem 4.1 in [H2])**.** *Suppose that conditions (M1)-(M5) hold with constants satisfying (19). Then the Markov operator P induced by the transition probability function of the chain Φ, given by (5), has the unique invariant probability measure μ_*^Φ such that $\mu_*^\Phi \in \mathcal{M}_{1,1}^V(X)$. Moreover, there exist $\bar{x} \in X$ and constants $c \in (0,\infty)$, $q \in [0,1)$ for which*

$$d_{\mathrm{FM}}(\mu P^n, \mu_*^\Phi) \le c\left(\int_X \rho_{\tilde{c}}(\bar{x},\cdot)\, d\left(\mu + \mu_*^\Phi\right) + 1\right) q^n \;\; \textit{for all } \; n \in \mathbb{N} \;\; \textit{and any} \;\; \mu \in \mathcal{M}_{1,1}^V(X).$$

The proof of Theorem 4.4 is based on applying the asymptotic coupling method introduced in [Hai02]. More precisely, we use [KS20, Theorem 2.1], which gives sufficient conditions for a general Markov chain (in terms of its Markovian coupling) to be exponentially ergodic in the sense described above.

Theorem 4.4 allows us to show the SLLN for the chain Φ. This can be done by appealing to a general result of A. Shirikyan [Shi03] (see Theorem 6.2 in [H2] for the precise formulation of the result which we apply in the proof of Theorem 4.5, and which is a modified version of [Shi03, Theorem 2.1]).

Theorem 4.5 (The SLLN for Φ; Theorem 4.3 in [H2])**.** *Suppose that conditions* (M1)–(M5) *hold with constants satisfying* (19)*. Then, for each $g \in \mathrm{Lip}_b(X)$ and any initial state $x \in X$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(Y_k, \xi_k) = \int_X g\, d\mu_*^{\Phi} \quad \mathbb{P}_x\text{- a.s.},$$

where μ_*^{Φ} is the unique invariant distribution for the Markov operator P (which exists by Theorem 4.4).

From now on we will assume that Θ, i.e. the set of indexes of the transformations $y \mapsto w_\theta(y)$, is endowed with a finite measure ϑ.

The next result concerns a one-to-one correspondence between invariant probability measures of the operator P (corresponding to the chain Φ) and invariant probability measures of the semigroup $\{P(t)\}_{t\in\mathbb{R}_+}$ (corresponding to the continuous-time process Ψ).

Theorem 4.6 (Theorem 4.4 in [H2])**.** *Let P, given by* (5)*, be the transition probability function of the chain Φ, and let $\{P(t)\}_{t\in\mathbb{R}_+}$ be the Markov semigroup corresponding to the PDMP Ψ, determined by* (1)*. Moreover, let the stochastic kernels $G, W : X \times \mathcal{B}(X) \to [0,1]$ be given by* (6) *and* (7)*, respectively.*

(1) If μ_*^{Φ} is an invariant probability measure for the Markov operator P, then $\mu_*^{\Psi} := \mu_*^{\Phi} G$ is an invariant probability measure for the Markov semigroup $\{P(t)\}_{t\in\mathbb{R}_+}$ and $\mu_*^{\Psi} W = \mu_*^{\Phi}$.

(2) If μ_*^{Ψ} is an invariant probability measure for the Markov semigroup $\{P(t)\}_{t\in\mathbb{R}_+}$, then $\mu_*^{\Phi} := \mu_*^{\Psi} W$ is an invariant probability measure for the Markov operator P and $\mu_*^{\Phi} G = \mu_*^{\Psi}$.

Combining Theorems 4.4 and 4.6 immediately gives the following:

Corollary 4.1 (Corollary 4.5 in [H2])**.** *Suppose that conditions* (M1)–(M5) *hold with constants satisfying* (19)*. Then the Markov semigroup $\{P(t)\}_{t\in\mathbb{R}_+}$ of the PDMP Ψ, determined by* (1)*, has the unique invariant probability measure.*

Theorems 4.5 and 4.6 then lead us to the SLLN for Ψ.

Theorem 4.7 (The SLLN for Ψ; Theorem 4.7 in [H2])**.** *Suppose that conditions* (M1)–(M5) *hold with a bounded (or, which is the same thing, constant) $\mathcal{L} : Y \to \mathbb{R}_+$, and that* (19) *is satisfied. Then, for any $g \in \mathrm{Lip}_b(X)$ and any initial state $(y,i) \in X$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t g\left(Y(s), \xi(s)\right)\, ds = \int_X g\, d\mu_*^{\Psi} \quad \mathbb{P}_{(y,i)}\text{-a.s.} \tag{20}$$

where μ_*^{Ψ} stands for the unique invariant distribution of Ψ (which exists by Corollary 4.1).

The additional assumption regarding the function $\mathcal{L}$ ensures that the operator $g \mapsto Gg$ preserves the Lipschitz continuity. This is necessary for our proof method to work, as it enables us to apply Theorem 4.5 for the Markov chain $\{Gg(Y_n, \xi_n)\}_{n\in\mathbb{N}_0}$. Obviously, the above-mentioned requirement is always fulfilled if the process evolves according to only one semiflow.

Section 5 in [H2] is intended as an attempt to demonstrate the generality of the abstract model. We show that it is flexible enough to capture at least two completely different dynamical systems, regarding a continuous-time model of prokaryotic gene expression (cf. [MTKY13]) and a discrete-time model for an autoregulated gene in bacterium (cf. [HHS16]). Although the first of them is spanned on a finite-dimensional space (and might possibly be analysed by using some simpler tools), we believe that the combination of these two examples shows the universality of the presented approach.

[H3] *Continuous dependence of an invariant measure on the jump rate of a PDMP*

Let us follow the notation already introduced above, when the results of [H1] and [H2] were presented. Moreover, let $\text{Lip}_b(Y)^*$ stand for the dual space of $(\text{Lip}_b(Y), \|\cdot\|_{\text{BL}})$, where $\|\cdot\|_{\text{BL}}$ is given by (17), with the operator norm $\|\cdot\|^*_{\text{BL}}$ given by

$$\|\varphi\|^*_{\text{BL}} := \sup\{|\varphi(f)| : \; f \in \text{Lip}_b(Y), \; \|f\|_{\text{BL}} \leq 1\} \quad \text{for any} \quad \varphi \in \text{Lip}_b(Y)^*.$$

According to [Dud66, Lemma 6]), the map $\mathcal{M}_{\text{sig}}(Y) \ni \mu \mapsto I_\mu \in \text{Lip}_b(Y)^*$ is injective, and thus the space $(\mathcal{M}_{\text{sig}}(X), \|\cdot\|_{\text{TV}})$ may be embedded into $(\text{Lip}_b(Y)^*, \|\cdot\|^*_{\text{BL}})$. This enables us to identify each measure $\mu \in \mathcal{M}_{\text{sig}}(Y)$ with the functional $I_\mu \in \text{Lip}_b(Y)^*$. In view of this, $\|\cdot\|^*_{\text{BL}}$ induces a norm in $\mathcal{M}_{\text{sig}}(Y)$, called the Fortet-Mourier (or bounded Lipschitz) norm $\|\cdot\|_{\text{FM}}$. Consequently, we can write

$$\|\mu\|_{\text{FM}} := \|I_\mu\|^*_{BL} = \sup\left\{\left|\int_Y f\,d\mu\right| : \; f \in \text{Lip}_b(Y), \; \|f\|_{\text{BL}} \leq 1\right\} \quad \text{for any} \quad \mu \in \mathcal{M}_{\text{sig}}(Y).$$

For any positive jump intensity λ, we study the version of the PDMP Ψ_λ, given by (1) and (2), whose deterministic behavior between random jumps is driven by just one semiflow S and whose jump mechanism is determined by the specific kernel J given by (8). The post-jump locations of Ψ_λ are then described by the chain Φ_λ with transition law P_λ defined as in (9).

The assumptions imposed on the model are very similar to those employed in [H2] (and so the assertion of Theorem 4.4 holds). Let us, however, state them explicitly below (any reader interested in their reasonableness is referred either to the previous section or to Section 3 of [H3]). We assume that there exist a point $\bar{y} \in Y$, a Borel measurable function $\mathcal{J} : Y \to [0, \infty)$ and constants $\alpha \in \mathbb{R}$, $L, L_w, L_p, \lambda_{\min}, \lambda_{\max}, \bar{p} > 0$, such that

$$LL_w + \frac{\alpha}{\lambda} < 1 \quad \text{for each} \quad \lambda \in [\lambda_{\min}, \lambda_{\max}], \tag{21}$$

and, for any $y_1, y_2 \in Y$, the following conditions hold:

$$\sup_{y\in Y}\int_0^\infty e^{-\lambda_{\min}t}\int_\Theta p_\theta\left(S(t,y)\right)\left\|w_\theta\left(S(t,\bar{y})\right)\right\|\,d\theta\,dt<\infty, \tag{$\Lambda1$}$$

$$\left\|S\left(t,y_1\right)-S\left(t,y_2\right)\right\|\leq Le^{\alpha t}\left\|y_1-y_2\right\| \quad \text{for} \quad t\in\mathbb{R}_+, \tag{$\Lambda2$}$$

$$\left\|S\left(t,y_1\right)-S\left(s,y_1\right)\right\|\leq (t-s)e^{\max\{\alpha s,\alpha t\}}\mathcal{J}\left(y_1\right) \quad \text{for} \quad 0\leq s\leq t, \tag{$\Lambda3$}$$

$$\int_\Theta p_\theta\left(y_1\right)\left\|w_\theta\left(y_1\right)-w_\theta\left(y_2\right)\right\|\,d\theta\leq L_w\left\|y_1-y_2\right\|, \tag{$\Lambda4$}$$

$$\int_\Theta \left|p_\theta\left(y_1\right)-p_\theta\left(y_2\right)\right|\,d\theta\leq L_p\left\|y_1-y_2\right\|, \tag{$\Lambda5$}$$

$$\int_{\Theta(y_1,y_2)}\min\left\{p_\theta\left(y_1\right),p_\theta\left(y_2\right)\right\}\,d\theta\geq\bar{p}, \tag{$\Lambda6$}$$

where $\Theta(y_1,y_2):=\{\theta\in\Theta:\ \|w_\theta(y_1)-w_\theta(y_2)\|\leq L_w\left\|y_1-y_2\right\|\}$.

Note that, upon assuming (21), we have $\lambda>\max\{0,\alpha\}$ for any $\lambda\in[\lambda_{\min},\lambda_{\max}]$. Moreover, let us introduce the subset $\mathcal{M}_{\mathrm{sig},\mathcal{J}}(Y)$ of the set $\mathcal{M}_{\mathrm{sig}}(Y)$ defined as

$$\mathcal{M}_{\mathrm{sig},\mathcal{J}}(Y)=\left\{\mu\in\mathcal{M}_{\mathrm{sig}}(Y):\ \int_Y\mathcal{J}\,d|\mu|<\infty\right\},\quad\text{where }\mathcal{J}\text{ is given in }(\Lambda3).$$

In Section 4 of [H3] we analyze certain properties of the Markov operator P_λ induced by the transition kernel P_λ of Φ_λ. In what follows, let $\Pi_{(t)}$ be given by (10) for any $t\in\mathbb{R}_+$.

Lemma 4.2 (Lemma 4.1 in [H3]). *Suppose that conditions* ($\Lambda3$)–($\Lambda5$) *hold. Then, for any* $\lambda>0$ *and any* $\mu\in\mathcal{M}_{\mathrm{sig},\mathcal{J}}(Y)$, *the function* $t\mapsto e^{-\lambda t}\mu\Pi_{(t)}$ *is Bochner integrable (all the necessary definitions and basic properties concerning Bochner integrals are gathered in Section 2 of* [H3]*) as a map from* $\mathbb{R}_+$ *to* $(\mathtt{Cl}(\mathcal{M}_{\mathrm{sig}}(Y)),\|\cdot\|^*_{\mathrm{BL}}|_{\mathtt{Cl}(\mathcal{M}_{\mathrm{sig}}(Y))})$, *and we have*

$$\mu P_\lambda=\int_0^\infty\lambda e^{-\lambda t}\mu\Pi_{(t)}\,dt.$$

Lemma 4.3 (Lemma 4.2 in [H3]). *Let* $f\in\mathrm{Lip}_b(Y)$. *Upon assuming* ($\Lambda2$), ($\Lambda4$) *and* ($\Lambda5$), *we have*

$$\left\|\mu\Pi_{(t)}\right\|_{\mathrm{FM}}\leq\left(1+\left(L_w+L_p\right)Le^{\alpha t}\right)\|\mu\|_{\mathrm{FM}}\quad\textit{for any}\quad\mu\in\mathcal{M}_{\mathrm{sig}}(Y),\ t\in\mathbb{R}_+.$$

Lemma 4.4 (Lemma 4.4 in [H3]). *Let* $\mathcal{M}_{\mathrm{sig}}(Y)$ *be endowed with the topology induced by the norm* $\|\cdot\|_{\mathrm{FM}}$, *and suppose that conditions* ($\Lambda2$)–($\Lambda5$) *hold. Then, the map*

$$(\max\{0,\alpha\},\infty)\times\mathcal{M}_{\mathrm{sig},\mathcal{J}}(Y)\ni(\lambda,\mu)\mapsto\mu P_\lambda\in\mathcal{M}_{\mathrm{sig}}(Y)$$

is jointly continuous.

Lemma 4.5 (Lemma 4.5 in [H3]). *Suppose that conditions* ($\Lambda1$), ($\Lambda2$) *and* ($\Lambda4$)–($\Lambda6$) *hold with constants satisfying* (21), *and, for any* $\lambda\in[\lambda_{min},\lambda_{max}]$, *let* $\mu_*^{\Phi_\lambda}$ *stand for the unique invariant probability measure of* Φ_λ *(existing due to Theorem 4.4). Then, the convergence*

$$\left\|\mu P_\lambda^n-\mu_*^{\Phi_\lambda}\right\|_{\mathrm{FM}}\to 0$$

is uniform with respect to λ, *whenever* $\mu\in\mathcal{M}_1(Y)$ *is such that* $\int_Y\|\cdot\|\,d\mu<\infty$.

Using the above lemmas, as well as [Rud76, Theorem 7.11], we obatin the first of the main results of the article [H3].

Theorem 4.8 (Theorem 5.2 in [H3])**.** *Suppose that conditions* ($\Lambda1$)–($\Lambda6$) *hold with constants satisfying* (21)*, and, for any* $\lambda \in [\lambda_{min}, \lambda_{max}]$*, let* $\mu_*^{\Phi_\lambda}$ *stand for the unique invariant probability measure of* Φ_λ *(existing due to Theorem 4.4). Then, for every* $\bar{\lambda} \in [\lambda_{\min}, \lambda_{\max}]$*, we have* $\mu_*^{\Phi_\lambda} \xrightarrow{w} \mu_*^{\Phi_{\bar{\lambda}}}$*, as* $\lambda \to \bar{\lambda}$ *(where the symbol "*$\xrightarrow{w}$*" denotes weak convergence od measures, that is,* $\lim_{\lambda\to\bar{\lambda}} \int_Y f\, d\mu_*^{\Phi_\lambda} = \int_Y f\, d\mu_*^{\Phi_{\bar{\lambda}}}$ *for any* $f \in \mathrm{C}_b(Y)$*).*

The second main result then follows from Theorems 4.8 and 4.6.

Theorem 4.9 (Theorem 5.3 in [H3])**.** *Let* ϑ *be a finite Borel measure on* Θ*. Further, suppose that conditions* ($\Lambda1$)–($\Lambda6$) *hold with constants satisfying* (21)*, and, for any* $\lambda \in [\lambda_{min}, \lambda_{max}]$*, let* $\mu_*^{\Psi_\lambda}$ *stand for the unique invariant probability measure of* Ψ_λ *(existing due to Corollary 4.1). Then, for any* $\bar{\lambda} \in [\lambda_{min}, \lambda_{max}]$*, we have* $\mu_*^{\Psi_\lambda} \xrightarrow{w} \mu_*^{\Psi_{\bar{\lambda}}}$*, as* $\lambda \to \bar{\lambda}$*.*

Thanks to the results presented in [H3] (Theorems 4.8 and 4.9 above), we can conclude that minor variations in the model parameter λ will have a negligible impact on the stationary distribution of this model. This characteristic ensures the suitability of the proposed abstract model for describing a wide range of real phenomena.

[H4] *A useful version of the central limit theorem for a general class of Markov chains*

Let (E, ρ) be a Polish metric space, and let d_{FM} denote the Fortet-Mourier distance, defined as in (18). The results of [H4] pertain to:

- **the criterion for the geometric mixing property (in d_{FM}) for a general class of Markov chains** (Section 2 in [H4]),
- the version of **the CLT for a general class of Markov chains** (Section 3 in [H4]),
- establishing **the CLT for an abstract model of gene expression introduced in** [H2] (Section 4 in [H4]).

In Section 2 of [H4], we draw on certain ideas used in [Hai02, Ś11, Cza18, KS20] and [D3]. The key results there are Lemmas 2.2 and 2.3, where the latter slightly strengthens the exponential mixing property (in d_{FM}) obtained and used in the proof of [KS20, Theorem 2.1]. Lemma 2.3 (which is a consequence of Lemma 2.2) is an essential tool in the proof of the main result of [H4], that is, Theorem 3.2.

Let $P : E \times \mathcal{B}(E) \to [0, 1]$ and $P : \mathcal{M}(E) \to \mathcal{M}(E)$ be a stochastic kernel and the Markov operator induced by this kernel, respectively. We impose the following conditions on P:

(B0) The Markov operator P has the Feller property.

(B1) There exist a Lyapunov function $V : E \to [0, \infty)$ and constants $a \in (0, 1)$ and $b \in (0, \infty)$ such that

$$PV(x) \le aV(x) + b \quad \text{for every} \quad x \in E.$$

Moreover, we require that there is a substochastic kernel $Q: E^2 \times \mathcal{B}(E^2) \to [0,1]$, satisfying

$$Q((x,y), A \times E) \le P(x,A), \quad Q((x,y), E \times A) \le P(y,A) \quad \text{for} \quad x,y \in E, \ A \in \mathcal{B}(E), \tag{22}$$

which enjoys the following properties:

(B2) There exist $F \subset E^2$ and $\delta \in (0,1)$ such that

$$\operatorname{supp} Q((x,y), \cdot) \subset F \quad \text{and} \quad \int_{E^2} \rho(u,v)\, Q((x,y), du \times dv) \le \delta \rho(x,y) \quad \text{for} \quad (x,y) \in F.$$

(B3) Letting $U(r) = \{(u,v) \in F : \rho(u,v) \le r\}$, $r > 0$, we have

$$\inf_{(x,y)\in F} Q\left((x,y), U\left(\delta\rho(x,y)\right)\right) > 0.$$

(B4) There exist constants $\beta \in (0,1]$ and $c_\beta > 0$ such that

$$Q\left((x,y), E^2\right) \ge 1 - c_\beta \rho^\beta(x,y) \quad \text{for every} \quad (x,y) \in F.$$

(B5) There exists a coupling $(\phi_n^{(1)}, \phi_n^{(2)})_{n\in\mathbb{N}_0}$ of P with transition law $C \ge Q$ (cf. Section 1.3 in [H4]) such that for some $\Gamma > 0$ and

$$K := \left\{(x,y) \in E^2 : \ (x,y) \in F \text{ and } V(x) + V(y) < \Gamma\right\}$$

we can choose $\gamma \in (0,1)$ and $c_\gamma > 0$, for which

$$\mathbb{E}_{x,y}(\gamma^{-\rho_K}) \le c_\gamma, \quad \text{whenever} \quad V(x) + V(y) < 4b(1-a)^{-1},$$

where

$$\rho_K = \inf\left\{n \in \mathbb{N} : \ (\phi_n^{(1)}, \phi_n^{(2)}) \in K\right\}$$

and $\mathbb{E}_{x,y}$ denotes the expectation operator w.r.t. $\mathbb{C}_{x,y}$, that is, the appropriate measure on $\mathcal{B}(E^2)$ for which

$$\mathbb{C}_{x,y}\left(\left(\phi_n^{(1)}, \phi_n^{(2)}\right) \in A \times E \,\middle|\, \phi_{n-1}^{(1)} = x\right) = P(x,A),$$
$$\mathbb{C}_{x,y}\left(\left(\phi_n^{(1)}, \phi_n^{(2)}\right) \in E \times A \,\middle|\, \phi_{n-1}^{(1)} = y\right) = P(y,A),$$

for any $A \in \mathcal{B}(E)$.

The desired lemma reads as follows:

Lemma 4.6 (Lemma 2.3 in [H4]). *Suppose that $P: E \times \mathcal{B}(E) \to [0,1]$ is a stochastic kernel such that conditions (B0)–(B5) hold with some substochastic kernel $Q: E^2 \times \mathcal{B}(E^2) \to [0,1]$ satisfying (22). Then there exist constants $q \in (0,1)$ and $c > 0$ such that*

$$\mathbb{E}_{x,y}\left|g\left(\phi_n^{(1)}\right) - g\left(\phi_n^{(2)}\right)\right| \le c\|g\|_{\mathrm{BL}}\, q^n (1 + V(x) + V(y)) \tag{23}$$

for all $(x,y) \in E^2$, $g \in \operatorname{Lip}_b(E)$ and $n \in \mathbb{N}_0$.

Let us now proceed to that part of article [H4] in which we establish the CLT for non-stationary Markov chains evolving in Polish spaces.

Assume that $\mu_* \in \mathcal{M}_1(E)$ is the unique invariant probability measure of P (which exists under the assumptions of [KS20, Theorem 2.1]). For any given Borel function $g : E \to \mathbb{R}$ such that $\int_E g^2\, d\mu_* < \infty$, we say that the CLT holds for $\{\bar{g}(\phi_n)\}_{n\in\mathbb{N}_0}$, where $\bar{g} = g - \int_E g\, d\mu_*$, if the value

$$\sigma^2(\bar{g}) = \lim_{n\to\infty} \mathbb{E}_{\mu_*}\left(\left(\frac{\bar{g}(\phi_1) + \ldots + \bar{g}(\phi_n)}{\sqrt{n}}\right)^2\right) \tag{24}$$

is finite and the averages $n^{-1/2}(\bar{g}(\phi_1) + \ldots + \bar{g}(\phi_n))$, $n \in \mathbb{N}$, converge in law to a centered normal random variable with variance $\sigma^2(\bar{g})$.

Now, let $D[0,1]$ denote the Skorochod space, i.e., the collection of all cádlág functions on $[0,1]$ (cf. [Bil99]). For any Borel function $g : E \to \mathbb{R}$, we introduce a process $\{B_n(g)\}_{n\in\mathbb{N}}$ with values in $D[0,1]$ by setting

$$B_n(g)(t) = \frac{1}{\sqrt{n}}\left(g(\phi_1) + \ldots + g\left(\phi_{\lceil nt\rceil}\right)\right), \quad 0 \le t < 1, \quad \text{and} \quad B_n(g)(1) = B_n(g)(1-)$$

for every $n \in \mathbb{N}$, where $\lceil a\rceil$ is the ceiling function of $a \in \mathbb{R}$. For any given Borel function $g : E \to \mathbb{R}$ such that $\int_E g^r\, d\mu_* < \infty$ for some $r > 2$, we say that $\{\bar{g}(\phi_n)\}_{n\in\mathbb{N}_0}$ satisfies *the Donsker invariance principle for the CLT* (the functional CLT), if $\sigma^2(\bar{g}) < \infty$ and $\{B_n(\bar{g})\}_{n\in\mathbb{N}}$ converges weakly to $\sigma(\bar{g})B$ in the space $D[0,1]$, where B is a standard Brownian motion on $[0,1]$.

Before we formulate the main theorem, we need to strengthen condition (B1) to the following form:

(B1)′ There exist a Lyapunov function $V : E \to [0,\infty)$ and constants $a \in (0,1)$ and $b \in (0,\infty)$ such that
$$PV^2(x) \le (aV(x) + b)^2 \quad \text{for every} \quad x \in E.$$

Obviously, due to the Hölder inequality, hypothesis (B1)′ implies (B1).

Theorem 4.10 (The CLT; Theorem 3.2 in [H4]). *Suppose that $P : E \times \mathcal{B}(E) \to [0,1]$ is a stochastic kernel such that conditions (B0)–(B5) with (B1) strengthened to (B1)′ hold with some substochastic kernel $Q : E^2 \times \mathcal{B}(E^2) \to [0,1]$ satisfying (22). Further, let $\{\phi_n\}_{n\in\mathbb{N}_0}$ be an E-valued time-homogeneous Markov chain with transition law P and initial distribution $\mu \in \mathcal{M}_{1,1}^V(E)$, where V is a Lyapunov function appearing in (B1)′. Then the CLT holds for $\{\bar{g}(\phi_n)\}_{n\in\mathbb{N}_0}$ whenever $g \in \mathrm{Lip}_b(E)$. Moreover, if the initial distribution of $\{\phi_n^*\}_{n\in\mathbb{N}_0}$ is equal to the unique invariant measure of P, then $\{\bar{g}(\phi_n^*)\}_{n\in\mathbb{N}_0}$ obeys the Donsker invariance principle for the CLT.*

An important step in the proof of Theorem 4.10 follows from [MW00, Corollary 1] and [MW00, Corollary 4] by M. Maxwell and M. Woodroofe (cf. Lemma 3.1 in [H4]).

Let us highlight that **a thorough analysis of the proof of Theorem 4.10 shows that its assertion remains valid under two more general (and simultaneously, much more abstract) hypotheses, namely:**

(i) conditions (B0) and either (B1) for both V and V^2 or (B1)′ are fulfilled;

(ii) there exists a Markovian coupling $\{(\phi_n^{(1)}, \phi_n^{(1)})\}_{n\in\mathbb{N}_0}$ of P for which condition (23) is satisfied.

In Section 4 of [H4] we indicate the usefulness of Theorem 4.10 by applying it to prove the CLT for the chain Φ examined in [H2]. Importantly, none of the existing versions of the CLT or the LIL, including those provided in [GHSZ19] and [BMS12], were applicable in this context.

[H5] *The Strassen invariance principle for certain non-stationary Markov-Feller chains*

The main results of [H5] are presented in Section 3, which is divided into two parts. The first – contains some general observations concerning martingales defined on the path space of a given ergodic Markov chain, while the second – presents the Strassen invariance principle for the LIL for the class of non-stationary Markov-Feller chains. We only outline the latter here.

Let $\mu \in \mathcal{M}_1(E)$ be chosen arbitrarily, and suppose that $P : E \times \mathcal{B}(E) \to [0,1]$ is a stochastic kernel satisfying conditions (B0) and (B1) with the Lyapunov function $V : E \to [0,\infty)$ of the form $V(x) = \rho(x, \bar{x})$ for $x \in E$, where $\bar{x}$ is an arbitrarily fixed point of E. In what follows, the chain governed by P with initial distribution μ will be denoted by $\{\phi_n\}_{n\in\mathbb{N}_0}$. Moreover, assume that there exists a substochastic kernel $Q : E^2 \times \mathcal{B}(E^2) \to [0,1]$ satisfying (22) such that conditions (B2)–(B5) hold for some $F \subset E^2$. Under these settings, P possesses the unique invariant probability measure μ_* such that $\mu_* \in \mathcal{M}_{1,1}^V$ (due to [KS20, Theorem 2.1]), and also, according to Lemma 4.6, condition (23) is fulfilled for some $q \in (0,1)$ and $c \in (0,\infty)$.

Now, let us define $\mathcal{C}$ as a Banach space of all real-valued continuous functions on $[0,1]$ with the supremum norm. By $\mathcal{K}$ we will denote the subspace of $\mathcal{C}$ consisting of all absolutely continuous functions f such that $f(0) = 0$ and $\int_0^1 (f'(t))^2\, dt \le 1$. Further, for any $g \in \mathrm{Lip}_b(E)$ and $\bar{g} = g - \int_E g\, d\mu_*$, we consider the sequence of random variables $\{r_n(\bar{g})\}_{n\in\mathbb{N}_0}$ with values in $\mathcal{C}$, determined by

$$\begin{aligned} r_n(\bar{g})(t) &:= \frac{\sum_{i=0}^{k-1} \bar{g}(\phi_i) + (nt-k)\bar{g}(\phi_k)}{\sigma(\bar{g})\sqrt{2n\ln(\ln(n))}} \quad \text{for} \quad n > e,\ t \in (0,1] \\ &\qquad\qquad \text{and } k \in \{1,\dots,n-1\} \text{ s.t. } k \le nt \le k+1, \\ r_n(\bar{g})(t) &:= 0 \quad \text{for} \quad n \le e \quad \text{or} \quad t = 0. \end{aligned} \tag{25}$$

For any given $g \in \mathrm{Lip}_b(E)$, we say that the Markov chain $\{\bar{g}(\phi_n)\}_{n\in\mathbb{N}_0}$ satisfies *the invariance principle for the LIL* if

$$0 < \sigma^2(\bar{g}) := \mathbb{E}_{\mu_*}\left(\left(\sum_{i=0}^{\infty} P^i \bar{g}(\phi_1) - \sum_{i=1}^{\infty} P^i \bar{g}(\phi_0)\right)^2\right) < \infty,$$

the family $\{r_n(\bar{g})\}_{n\in\mathbb{N}_0}$ is relatively compact in $\mathcal{C}$, and the set of its limit points coincides with $\mathcal{K}$ $\mathbb{P}_\mu$-a.s.

The main result of [H5] is formulated in the same spirit as [KS20, Theorem 2.1] and Theorem 3.2 in [H4]. While hypotheses (B0)–(B5), together with (19), are sufficient for the Markov operator P to be geometrically ergodic in d_{FM}, our proof of the Strassen invariance principle for the LIL additionally requires the following condition:

(B1*) There exist $a^* \in (0,1)$ and $b^* \in (0,\infty)$ such that, for any $\nu \in \mathcal{M}^V_{1,2+r}(E)$,

$$\left(\int_E V^{2+r}\,d(P\nu)\right)^{1/(2+r)} \le a^* \left(\int_E V^{2+r}\,d\nu\right)^{1/(2+r)} + b^*.$$

At this point, let us briefly compare condition (B1*) with (B1′), which has been employed in [H4] to establish the CLT, and which is a stronger version of (B1). Condition (B1*) is of the same type, although, in general, it does not need to imply (B1). Consequently, in Theorem 4.7 in [H5] we assume both (B1) and (B1*).

Theorem 4.11 (*The Strassen invariance principle* for the LIL; Theorem 4.7 in [H5])**.** *Suppose that $\{\phi_n\}_{n\in\mathbb{N}_0}$ is an E-valued time-homogeneous Markov chain with transition law P and initial distribution μ such that $\mu \in \mathcal{M}^V_{1,2+r}(E)$ for some $r \in (0,2)$. Further, assume that there exists a substochastic kernel $Q : E^2 \times \mathcal{B}(E^2) \to [0,1]$ satisfying (22), such that conditions (B0)–(B5) and (B1*) hold for P and Q with some $F \subset E^2$. Then, for any $g \in \mathrm{Lip}_b(E)$ such that $\sigma^2(\bar{g}) > 0$ (e.g., $g \in \mathrm{Lip}_b(E)$ that is not constant μ_*-a.e.; the finiteness of $\sigma^2(\bar{g})$ is established in Lemmas 4.9 and 4.10 in [H5]), the chain $\{\bar{g}(\phi_n)\}_{n\in\mathbb{N}_0}$ obeys the Strassen invariance principle for the LIL.*

The techniques that we use to prove Theorem 4.11 are mainly based on those employed in [BMS12] and [D1], which, in turn, refer to the martingale result by C.C. Heyde and D.J. Scott [HS73].

Analyzing the proof of Theorem 4.11 shows that its assertion remains valid under two more general (and simultaneously, much more abstract) hypotheses, namely:

(i) condition (B0) and (B1*) are fulfilled;

(ii) there exists a Markovian coupling $\{(\phi_n^{(1)}, \phi_n^{(2)})\}_{n\in\mathbb{N}_0}$ of P for which condition (23) is satisfied.

It is, however, worth noting that, rather than verifying condition (ii) directly, it is often much easier to verify the assumptions of Theorem 4.11 (or those of Theorem 4.10). In fact, for explicitly defined random dynamical systems (such as those in [H2] or [KS20] concerning a random IFS with an arbitrary set of transformations), it is quite intuitive how to define Q (cf. Eq. (6.9) in [H2] or the proof of [KS20, Proposition 3.1])).

The functional LIL for the chain Φ, studied in [H2], is established in Section 5 of [H5].

[H6] ***The law of the iterated logarithm for random interval homeomorphisms***

Let $(f_1,\ldots,f_N;p_1,\ldots,p_N)$ be an admissible IFS. Moreover, let $\Sigma=\{1,\ldots,N\}^{\mathbb{N}}$ be equipped with the product topology induced by the discrete topology on $\{1,\ldots,N\}$. For every $n\in\mathbb{N}$, define

$$f_\omega^n = f_{\omega_n}\circ\ldots\circ f_{\omega_1} = f_{(\omega_1,\ldots,\omega_n)} \quad \text{for any} \quad \omega=(\omega_1,\omega_2,\ldots)\in\Sigma.$$

By $\mathbb{P}$ we denote the measure on Σ, which is the product measure of the probability vector $(p_1,\ldots,p_N)$. By abuse of notation, we shall also write $\mathbb{P}$ for the product measure of the probability vector $(p_1,\ldots,p_N)$ on $\Sigma_n=\{1,\ldots,N\}^n$ for $n\in\mathbb{N}$.

Now, let $\{X_n\}_{n\in\mathbb{N}_0}$ be the Markov chain, corresponding to the admissible IFS $(f_1,\ldots,f_N;p_1,\ldots,p_N)$, with transition probability function P, given by (15), defined on the coordinate space $([0,1]^{\mathbb{N}},\mathcal{B}([0,1]^{\mathbb{N}}))$, equipped with an appropriate collection $\{\mathbb{P}_\nu\}_\nu$ of probability measures on $\mathcal{B}([0,1]^{\mathbb{N}})$ such that $\mathbb{P}_\nu(X_{n+1}\in A|X_n=x)=P(x,A)$ and $\mathbb{P}_\nu(X_0\in A)=\nu(A)$ for any probability measure ν on $[0,1]$. For $\nu=\delta_x$, $x\in[0,1]$, let us simply write $\mathbb{P}_x$.

Note that for $n\in\mathbb{N}$ and $A_1,\ldots,A_n\in\mathcal{B}([0,1])$ we have

$$\begin{aligned}
&\mathbb{P}_x\left((X_1,\ldots,X_n)\in A_1\times\ldots\times A_n\right)\\
&\qquad= \sum_{(\omega_1,\ldots,\omega_n)\in\Sigma_n} \mathbb{1}_{A_1\times\ldots\times A_n}\left(f_{\omega_1}(x),\ldots,f_{(\omega_1,\ldots,\omega_n)}(x)\right) p_{\omega_1}\ldots p_{\omega_n}\\
&\qquad= \int_{\Sigma_n} \mathbb{1}_{A_1\times\ldots\times A_n}\left(f_{\omega_1}(x),\ldots,f_{(\omega_1,\ldots,\omega_n)}(x)\right)\mathbb{P}\left(d\omega_1\times\ldots\times d\omega_n\right)\\
&\qquad= \int_{\Sigma} \mathbb{1}_{A_1\times\ldots\times A_n}\left(f^1_\omega(x),\ldots,f^n_\omega(x)\right)\mathbb{P}\left(d\omega\right).
\end{aligned}$$

By [CS20, Theorems 1 and 2] we know that the regular Markov operator P (induced by (15)) has the unique invariant probability measure μ_* on $(0,1)$. In the case where the initial distribution of $\{X_n\}_{n\in\mathbb{N}_0}$ is its stationary distribution μ_*, we will write $\{X_n^*\}_{n\in\mathbb{N}_0}$.

The two main results of [H6] read as follows:

Proposition 4.1 (The annealed LIL; Proposition in [H6]). *If $\varphi:[0,1]\to\mathbb{R}$ is a Lipschitz function satisfying $\int_{[0,1]}\varphi\,d\mu_*=0$, then there exists a constant $\sigma\in[0,\infty)$ such that*

$$\limsup_{n\to\infty}\frac{\varphi(X_1^*)+\ldots+\varphi(X_n^*)}{\sqrt{2n\ln(\ln(n))}}=\sigma \quad \mathbb{P}_{\mu_*}\text{-a.s.}$$

Theorem 4.12 (The quenched LIL; Theorem in [H6]). *If $\varphi:[0,1]\to\mathbb{R}$ is a Lipschitz function satisfying $\int_{[0,1]}\varphi\,d\mu_*=0$, then there exists a constant $\sigma\in[0,\infty)$ such that for every $x\in(0,1)$*

$$\limsup_{n\to\infty}\frac{\varphi\left(f_x^1(x)\right)+\ldots+\varphi\left(f_x^n(x)\right)}{\sqrt{2n\ln(\ln(n))}}=\sigma \quad \mathbb{P}\text{-a.s.}$$

Given the extensive body of research by many mathematicians on the model discussed in [H6], the importance of a fundamental result such as the LIL shall be properly appreciated.

References

[AHVG15] T. Alkurdi, S.C. Hille, and O. Van Gaans. Persistence of stability for equilibria of map iterations in Banach spaces under small perturbations. *Potential Anal.*, 42(11):175–201, 2015.

[AM14] L. Alsedá and M. Misiurewicz. Random interval homeomorphisms. *Publ. Mat.*, 58:15–36, 2014.

[AV10] A. Avila and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Invent. Math.*, 181:115–178, 2010.

[BD85] M.F. Barnsley and S.G. Demko. Iterated function systems and the global construction of fractals. *Proc. Roy. Soc. London, Ser. A*, 399(1817):243–275, 1985.

[BDEG88] M.F. Barnsley, S.G. Demko, J.H. Elton, and J.S. Geronimo. Invariant measures for Markov processes arising from iterated function systems with place dependent probabilities. *Ann. Inst. H. Poincaré*, 24(3):367–394, 1988.

[BH12] Y. Bakhtin and T. Hurth. Invariant densities for dynamical systems with random switching. *Nonlinearity*, 25(10):2937–2952, 2012.

[BHS18] M. Benaïm, T. Hurth, and E. Strickler. A user-friendly condition for exponential ergodicity in randomly switched environments. *Electron. Commun. Probab.*, 23:1–12, 2018.

[Bil99] P. Billingsley. *Convergence of probability measures.* John Wiley & Sons, Inc., New York, 1999.

[Bin86] N. Bingham. Variants on the law of the iterated logarithm. *Lond. Math. Soc.*, 18(5):433–467, 1986.

[BKKP05] O. Boxma, H. Kaspi, O. Kella, and D. Perry. On/off storage systems with state-dependent inpout. *Probab. Engrg. Inform. Sci.*, 19(1):1–14, 2005.

[BL16] M. Benaïm and C. Lobry. Lotka Volterra with randomly fluctuationg environments or 'how switching between benefcial environments can make survival harder'. *Ann. Appl. Probab.*, 26(6):3754–3785, 2016.

[BLBMZ12] M. Benaïm, S. Le Borgne, F. Malrieu, and P.-A. Zitt. Quantitative ergodicity for some switched dynamical systems. *Electron. Commun. Probab.*, 17(0):1–14, 2012.

[BLBMZ15] M. Benaïm, S. Le Borgne, F. Malrieu, and P.-A. Zitt. Qualitative properties of certain piecewise deterministic Markov processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 51(3):1040–1075, 2015.

[BMS12] W. Bołt, A.A. Majewski, and T. Szarek. An invariance principle for the law of the iterated logarithm for some Markov chains. *Studia Math.*, 212:41–53, 2012.

[Bog07] V.I. Bogachev. *Measure theory*, volume II. Springer-Verlag, Berlin, 2007.

[BS19] M. Benaïm and E. Strickler. Random switching between vector fields having a common zero. *Ann. Appl. Probab.*, 29(1):326–375, 2019.

[BTK16] W. Biedrzycka and M. Tyran-Kamińska. Existence of invariant densities for semiflows with jumps. *J. Math. Anal. Appl.*, 435(1):61–84, 2016.

[CD99] O.L.V. Costa and F. Dufour. Stability of piecewise deterministic Markov processes. *SIAM J. Control Optim.*, 37(5):1483–1502, 1999.

[CD08] O.L.V. Costa and F. Dufour. Stability and ergodicity of piecewise deterministic Markov processes. In *2008 47th IEEE Conference on Decision and Control.* IEEE, 2008.

[CDMR12] A. Crudu, A. Debussche, A. Muller, and O. Radulescu. Convergence of stochastic gene networks to hybrid piecewise deterministic processes. *Ann. Appl. Probab.*, 22(5):1822–1859, 2012.

[CH14] D. Czapla and K. Horbacz. Equicontinuity and stability properties of Markov chains arising from iterated function systems on Polish spaces. *Stoch. Anal. Appl.*, 32(1):1–29, 2014.

[CH15] B. Cloez and M. Hairer. Exponential ergodicity for Markov processes with random switching. *Bernoulli*, 21(1):505 – 536, 2015.

[CK19] D. Czapla and J. Kubieniec. Exponential ergodicity of some Markov dynamical systems with application to a Poisson driven stochastic differential equation. *Dyn. Syst.*, 34(1):130–156, 2019.

[Cos90] O.L.V. Costa. Stationary distributions for piecewise-deterministic Markov processes. *J. Appl. Prob.*, 27(1):60–73, 1990.

[CS20] K. Czudek and T. Szarek. Ergodicity and central limit theorem for random interval homeomorphisms. *Isr. J. Math.*, 239:75–98, 2020.

[Cza12] D. Czapla. A criterion of asymptotic stability for Markov-Feller e-chains on Polish spaces. *Ann. Polon. Math.*, 105(3):267–291, 2012.

[Cza18] D. Czapla. A criterion on asymptotic stability for partially equicontinuous Markov operators. *Stoch. Process Their. Appl.*, 128(11):3656–3678, 2018.

[Dav84] M.H.A. Davis. Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. *J. Roy. Statist. Soc. Ser. B*, 46(3):353–388, 1984.

[Dav93] M.H.A. Davis. *Markov models and optimization.* Chapman and Hall, London, 1993.

[Dav06] M.H.A. Davis. *Point process theory and applications: Marked point and piecewise deterministic processes.* Birkhäuser-Verlag, Boston, 2006.

[DF99] P. Diaconis and D. Freedman. Iterated random functions. *SIAM Rev.*, 41(1):45–76, 1999.

[DHT84] O. Diekmann, H.J. Heijmans, and H.R. Thieme. On the stability of the cells size distribution. *J. Math. Biol.*, 19:227–248, 1984.

[DL01] Y. Derriennic and M. Lin. The central limit theorem for Markov chains with normal transition operators, started at a point. *Probab. Theory Related Fields*, 119(4):508–528, 2001.

[DMS14] D. Douc, E. Moulines, and D. Stoffer. *Nonlinear time series: theory, methods and applications with R examples.* Chapman and Hall/CRC, New York, 2014.

[DU77] J. Diestel and J.J. Uhl, Jr. *Vector measures.* American Mathematical Society, Mathematical Surveys, vol. 15, Providence, Rhode Island, 1977.

[Dud66] R. Dudley. Convergence of Baire measures. *Studia Math.*, 27:251–268, 1966.

[Dyn65] E.B. Dynkin. *Markov processes I.* Springer-Verlag, Berlin, 1965.

[Dyn00] E.B. Dynkin. *Selected papers of E.B. Dynkin with commentary.* American Mathematical Society, RI; International Press, Cambridge, MA, 2000.

[EHM15] J.H.M. Evers, S.C. Hille, and A. Muntean. Mild solutions to a measure-valued mass evolution problem with flux boundary conditions. *J. Differ. Equ.*, 259(3):1068–1097, 2015.

[EK86] N.E. Ethier and T.G. Kurtz. *Markov processes. Characterization and convergence.* John Wiley & Sons, Inc., Hoboken, New Jersey, 1986.

[GH17] M. Gharaei and A.J. Homburg. Random interval diffeomorphisms. *Discrete Contin. Dyn. Syst. - S.*, 10(2):241—-272, 2017.

[GHLSG19] P. Gwiazda, S.C. Hille, K. Łyczek, and A. Świerczewska-Gwiazda. Differentiability in perturbation parameter of measure solutions to perturbed transport equation. *Kinet. Relat. Mod.*, 12(5):1093–1108, 2019.

[GHSZ19] J. Gulgowski, S.C. Hille, T. Szarek, and M.A. Ziemlańska. Central limit theorem for some non-stationary Markov chains. *Studia Math.*, 246:109–131, 2019.

[GL78] M.I. Gordin and B.A. Lifšic. Central limit theorem for stationary Markov processes. *Dokl. Akad. Nauk SSSR*, 239(4):766–767, 1978.

[GL81] M.I. Gordin and B.A. Lifšic. A remark about a Markov process with normal transition operator. *Third Vilnius Conf. Proba. Stat., Akad. Nauk Litovsk, Vilnius*, 1:147–148, 1981.

[GR09] C. Graham and P. Robert. Interacting multi-class transmissions in large stochastic networks. *Ann. Appl. Probab.*, 19(6):2334–2361, 2009.

[Hai02] M. Hairer. Exponential mixing properties of stochastic PDEs through asymptotic coupling. *Probab. Theory Related Fields*, 124, 2002.

[HH80] P. Hall and C.C. Heyde. *Martingale limit theory and its application.* Elsevier, New York, 1980.

[HHS16] S.C. Hille, K. Horbacz, and T. Szarek. Existence of a unique invariant measure for a class of equicontinuous Markov operators with application to a stochastic model for an autoregulated gene. *Ann. Math. Blaise Pascal*, 23(2):171–217, 2016.

[HM08] M. Hairer and J.C. Mattingly. Spectral gaps in Wasserstein distances and the 2D stochastic Navier–Stokes equations. *Ann. Probab.*, 36(6):2050–2091, 2008.

[HM11] M. Hairer and J.C. Mattingly. *Yet another look at Harris' ergodic theorem for Markov chains. In: Dalang, R., Dozzi, M., Russo, F. (eds) Seminar on Stochastic Analysis, Random Fields and Applications VI. Progress in Probability, vol. 63.* Springer, Basel, 2011.

[Hol05] H. Holzmann. The central limit theorem for stationary Markov processes with normal generator – with applications to hypergroups. *Stochastics*, 77(4):371–380, 2005.

[Hor02] K. Horbacz. Invariant measures related with randomly connected Poisson driven differential equations. *Ann. Polon. Math.*, 79(1):31–44, 2002.

[Hor08] K. Horbacz. Invariant measures for random dynamical systems. *Dissertationes Math.*, 451:1–63, 2008.

[Hor16] K. Horbacz. The central limit theorem for random dynamical systems. *J. Stat. Phys.*, 164(6):1261–1291, 2016.

[HS73] C.C. Heyde and D.J. Scott. Invariance principles for the law of the iterated logarithm for martingales and processes with stationary increments. *Ann. Probab.*, 1(3):428–436, 1973.

[HS06] K. Horbacz and T. Szarek. Irreducible Markov systems on Polish spaces. *Studia Math.*, 177(3):285–295, 2006.

[HS16] K. Horbacz and M. Ślęczka. Law of large numbers for random dynamical systems. *J. Stat. Phys.*, 162:671–684, 2016.

[HSWZ21] S.C. Hille, T. Szarek, D. Worm, and M.A. Ziemlańska. Equivalence of equicontinuity concepts for Markov operators derived from a Schur-like property for spaces of measures. *Stat. Probab. Lett.*, 169(108964), 2021.

[HSZ17] S.C. Hille, T. Szarek, and M. Ziemlańska. Equicontinuous families of Markov operators in view of asymptotic stability. *C. R. Math.*, 355(12):1247–1251, 2017.

[IK02] K. Ito and F. Kappel. *Evolution equations and approximations, vol. 61 of Ser. Adv. Math. Appl. Sci.* World Scientific, New Jersey, 2002.

[Jam64] B. Jamison. Asymptotic behaviour of succesive iterates of continuous functions under a Markov operator. *J. Math. Anal. Appl.*, 9(2):203–214, 1964.

[JT20] R. Jin and A. Tan. Central limit theorems for Markov chains based on their convergence rates in Wasserstein distance. *Preprint available at* https://arxiv.org/abs/2002.09427, 2020.

[Kaz13] J. Kazak. Piecewise-deterministic Markov processes. *Ann. Polon. Math.*, 109(3):279–296, 2013.

[Kle13] A. Klenke. *Probability theory: a comprehensive course.* Springer, London, 2013.

[KPS10] T. Komorowski, S. Peszat, and T. Szarek. On ergodicity of some Markov processes. *Ann. of Probab.*, 38(4):1401–1443, 2010.

[KPS13] T. Komorowski, S. Peszat, and T. Szarek. Passive tracer in a flow corresponding to two-dimensional stochastic Navier-Stokes equations. *Nonlinearity*, 26(7):1999–2026, 2013.

[KS11] R. Kapica and M. Szarek, T. Ślęczka. On a unique ergodicity of some Markov processes. *Potential Anal.*, 36(4):589–606, 2011.

[KS14] T. Komorowski and T. Szarek. The law of the iterated logarithm for passive tracer in a two-dimensional flow. *J. London Math. Soc.*, 89(2):482–498, 2014.

[KS20] R. Kapica and M. Ślęczka. Random iteration with place dependent probabilities. *Probab. Math. Statist.*, 40(1):119–137, 2020.

[KV86] C. Kipnis and S.R.S. Varadhan. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.*, 104:1–19, 1986.

[KW12] T. Komorowski and A. Walczuk. Central limit theorem for Markov processes with spectral gap in the Wasserstein metric. *Stoch. Proc. Appl.*, 122(5):2155–2184, 2012.

[Las95] A. Lasota. From fractals to stochastic differential equations. In P. Garbaczewski, M. Wolf, and A. Weron, editors, *Chaos – The interplay between stochastic and deterministic behaviour (Proceedings of the XXXIst Winter School of Theoretical Physics, Karpacz, Poland, 13-24 Feb 1995)*, volume 457 of *Lecture Notes in Physics*, pages 235–255, Berlin, 1995. Lecture Notes in Physics 457, Springer Verlag.

[Lév35] P. Lévy. Propriétés asymptotiques des sommes de variables indépendantes ou enchainées. *Journal des mathématiques pures et appliquées. Series 9.*, 14(4):347–402, 1935.

[Lim00] W. Liming. Functional law of iterated logarithm for additive functionals of reversible Markov processes. *Acta Math. Appl. Sin.*, 16(2):149–161, 2000.

[Lin02] T. Lindvall. *Lectures on the coupling method.* Dover Publications, Inc., Mineola, New York, 2002.

[LL23] Y. Liu and Z. Liu. Relation between the eventual continuity and the e-property for Markov-Feller semigroups. *available as a preprint at https://arxiv.org/abs/2302.08755*, 2023.

[LM94] A. Lasota and J. Myjak. Generic properties of fractal measures. *Bull. Polish Acad. Sci. Math.*, 42(4):283–296, 1994.

[LM98] A. Lasota and J. Myjak. Semifractals on Polish spaces. *Bull. Pol. Ac. Math.*, 46(2):179–196, 1998.

[LM99] A. Lasota and M.C. Mackey. Cell division and the stability of cellular populations. *J. Math. Biol.*, 38(3):241–261, 1999.

[Löc18] E. Löcherbach. Absolute continuity of the invariant measure in piecewise deterministic Markov Processes having degenerate jumps. *Stoch. Process. their Appl.*, 128(6):1797–1829, 2018.

[LS05] A.N. Lagerås and Ö. Stenflo. Central limit theorems for contractive Markov chains. *Nonlinearity*, 18:1955–1965, 2005.

[LS06] A. Lasota and T. Szarek. Lower bound technique in the theory of a stochastic differential equation. *J. Differ. Equ.*, 231(2):513–533, 2006.

[LT03] A. Lasota and J. Traple. Invariant measures related with Poisson driven stochastic differential equation. *Stoch. Proc. Appl.*, 106(1):81–93, 2003.

[LY94] A. Lasota and J. Yorke. Lower bound technique for Markov operators and iterated function systems. *Rand. Comput. Dynam.*, 2(1):41–77, 1994.

[Mal17] D. Malicet. Random walks on Homeo(S^1). *Commun. Math. Phys.*, 356:1083–1116, 2017.

[Mat02] J.C. Mattingly. Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics. *Commun. Math.*, 230:421–462, 2002.

[MT93a] S.P. Meyn and R. Tweedie. Criteria for stability of Markovian processes III: Foster–Lyapunov criteria for continuous time processes, with examples. *Adv. in Appl. Probab.*, 25(3):518–548, 1993.

[MT93b] S.P. Meyn and R.L. Tweedie. *Markov chains and stochastic stability.* Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[MTKY13] M.C. Mackey, M. Tyran-Kamińska, and R. Yvinec. Dynamic behavior of stochastic gene expression models in the presence of bursting. *SIAM J. Appl. Math.*, 73(5):1830–1852, 2013.

[MW00] M. Maxwell and M. Woodroofe. Central limit theorems for additive functionals of Markov chains. *Ann. Probab.*, 28:713–724, 2000.

[MZ10] C. Marinelli and G. Ziglio. Ergodicity for nonlinear stochastic evolution equations with multiplicative Poisson noise. *Dyn. Partial Differ. Equ.*, 7(1):1–23, 2010.

[OLK12] S. Olla, C. Landim, and T. Komorowski. *Fluctuations in Markov processes. Time symmetry and martingale approximation.* Springer, Berlin, Heidelberg, 2012.

[Pic19] B. Piccoli. Measure differential equations. *Arch. Ration. Mech. Anal.*, 233(3):1289–1317, 2019.

[RTK17] R. Rudnicki and M. Tyran-Kamińska. *Piecewise deterministic processes in biological models.* SpringerBr. Appl. Sci. Technol., Cham, 2017.

[RTW12] M.G. Riedler, M. Thieullen, and G. Wainrib. Limit theorems for infinite-dimensional piecewise deterministic Markov processes. Applications to stochastic excitable membrane models. *Electron. J. Probab.*, 17(55):1–48, 2012.

[Rud76] W. Rudin. *Principles of mathematical analysis.* McGraw-Hill, Inc., New York, 1976.

[Ś11] M. Ślęczka. The rate of convergence for iterated function systems. *Studia Math.*, 205:201–214, 2011.

[Shi03] A. Shirikyan. A version of the law of large numbers and applications. *In: Probabilistic Methods in Fluids, Proceedings of the Swansea Workshop held on 14 - 19 April 2002, World Scientific, New Jersey*, pages 263–271, 2003.

[Sny75] D. Snyder. *Random point processes.* Wiley, New York, 1975.

[SSU10] T. Szarek, M. Ślęczka, and M. Urbański. On stability of velocity vectors for some passive tracer models. *Bull. London Math. Soc.*, 42(5):923–936, 2010.

[Ste94] Ł. Stettner. Remarks on ergodic conditions for Markov processes on Polish spaces. *Bull. Polish Acad. Sci. Math.*, 42:103–114, 1994.

[Ste01] Ö. Stenflo. A note on a theorem of Karlin. *Stat. Probab. Lett.*, 54(2):183–187, 2001.

[SW11] T. Szarek and D.T.H. Worm. Ergodic measures of Markov semigroups with the e-property. *Ergod. Theory Dyn. Syst.*, 32(3):1117–1135, 2011.

[Sza03] T. Szarek. Invariant measures for nonexpansive Markov operators on Polish spaces. *Dissertationes Math.*, 415:1–62, 2003.

[Sza06] T. Szarek. Feller processes on nonlocally compact spaces. *Ann. Probab.*, 34(5):1849–1863, 2006.

[Sza13] T. Szarek. Lower bound technique and its applications to function systems and stochastic partial differential equations. *Mat. Stos.*, 41(2):185–198, 2013.

[Wer05] I. Werner. Contractive Markov systems. *J. London Math. Soc.*, 71(1):236–258, 2005.

[Wor10] D.T.H. Worm. *Semigroups on spaces of measures*. PhD. thesis, Mathematical Institute, Faculty of Science, Leiden University, The Netherlands, available at: https://math.leidenuniv.nl/scripties/WormThesis.pdf, 2010.

[WW18] S. Wędrychowicz and A. Wiśnicki. On some results on the stability of Markov operators. *Studia Math.*, 241(1):41–55, 2018.

[ZW08] O. Zhao and M. Woodroofe. Law of the iterated logarithm for stationary processes. *Ann. Probab.*, 36(1):127 – 142, 2008.

4.4 Description of the applicant's contribution to each work comprising the achievement, set out in art. 219 para 1. point 2 of the Act

Article [H1]

R. Kukulski, H. Wojewódka-Ściążko,
***The e-property of asymptotically stable Markov-Feller operators*,**
Colloq. Math. 165 (2021), 269–283

The results were established while my co-author R. Kukulski was preparing his master's thesis under my supervision. I suggested the topic of the work, and R. Kukulski provided examples, described in Section 4 of the article. We both contributed equally to proving the main theorems. The initial draft of the paper was written by R. Kukulski and later revised by me. I added Section 1 and made significant corrections in the subsequent sections.

Article [H2]

D. Czapla, K. Horbacz, H. Wojewódka-Ściążko,
***Ergodic properties of some piecewise deterministic Markov process with application to gene expression modelling*,**
Stoch. Proc. Appl. 130 (2020), no. 5, 2851–2885

K. Horbacz initiated this project and proposed the overall structure of the model under consideration. Drawing from my PhD expertise, I was able to thoroughly understand the proof of the criterion by R. Kapica and M. Ślęczka [KS20, Theorem 2.1]. As a consequence, I was able to provide assistance to D. Czapla in establishing one of our main results, Theorem 4.1, whose proof extensively relies on the utilization of this criterion. I also made significant contributions to the proofs of Theorems 4.2–4.7. D. Czapla originated Section 3, addressing the validity of the assumptions made. K. Horbacz suggested Example 5.1, describing prokaryotic (bacterial) gene expression, while I proposed Example 5.2, which pertains to auto-regulated gene behaviour in bacteria. After consulting with S.C. Hille, I enhanced the biological interpretation in Example 5.1, emphasizing that genes in prokaryotes can (and frequently are) organised in the so-called *operons*. D. Czapla initially drafted the article, which was subsequently reviewed and completed by K. Horbacz and myself. I am the author of the Introduction.

Article [H3]

D. Czapla, S.C. Hille, K. Horbacz, H. Wojewódka-Ściążko,
***Continuous dependence of an invariant measure on the jump rate of a piecewise deterministic Markov process*, Math. Biosci. Eng. 17 (2020), no. 2, 1059–1073**

We obtained the results described in the article while I was carrying out my research project *Mathematical Modelling with Measures* funded by the National Science Centre under the registration number 2018/02/X/ST1/01518. This project provided me with the opportunity to conduct a research visit to the Mathematical Institute at Leiden University in December 2018. The main ideas emerged during my stay in Leiden, where I collaborated with S.C. Hille. Together, we formulated the research hypothesis and prepared the proof outline. Upon my return, in collaboration with D. Czapla and K. Horbacz, I refined all the details and subsequently wrote the article (with the assistance of D. Czapla).

Article [H4]

D. Czapla, K. Horbacz, H. Wojewódka-Ściążko,
***A useful version of the central limit theorem for a general class of Markov chains*, J. Math. Anal. Appl. 484 (2020), no. 1, 123725**

I was the project's main coordinator and driving force. The sketch of the proof of the main result of this paper was proposed by me. Drawing on my previous experience, I noticed that the desired version of the central limit theorem can be proven by creatively applying the asymptotic coupling techniques, introduced and developed by M. Hairer (since 2002). The lemma essential to the proof of this theorem was demonstrated by D. Czapla (with my assistance). I wrote the article, while D. Czapla and K. Horbacz made the necessary revisions to the proofs and performed some text editing.

Article [H5]

D. Czapla, K. Horbacz, H. Wojewódka-Ściążko,
***The Strassen invariance principle for certain nonstationary Markov-Feller chains*, Asymptot. Anal. 121 (2021), no. 1, 1–34**

The core ideas presented in this paper were conceived by me. The inability to directly apply a version of the law of the iterated logarithm by W. Bołt, A.A. Majewski and T. Szarek [BMS12] to a wide class of random dynamical systems, which we investigated in our previous papers and which are applicable, e.g., in molecular biology, motivated me to establish a new criterion. It was my suggestion to utilize certain techniques based on the construction of an asymptotic Markovian coupling in the proof, in addition to employing a martingale method. I initially drafted the paper, which was then reviewed by D. Czapla and K. Horbacz. D. Czapla completed certain proofs, particularly those pertaining to several general observations concerning martingales (included in the Appendix).

Article [H6]

K. Czudek, T. Szarek, H. Wojewódka-Ściążko,
***The law of the iterated logarithm for random interval homeomorphisms*, Isr. J. Math. 246 (2021), 47–53**

The work was conducted during the period when K. Czudek was preparing his doctoral dissertation under the supervision of T. Szarek. After establishing the central limit theorem for certain admissible iterated function systems [CS20], K. Czudek and T. Szarek aimed to further prove the law of the iterated logarithm for them. Leveraging my experience and knowledge of the literature on the law of the iterated logarithm, I suggested (during a scientific discussion with T. Szarek) that the appropriate utilization of the criterion by O. Zhao and M. Woodroofe [ZW08] could lead to the desired result. We jointly verified and confirmed my idea. Importantly, K. Czudek provided justification for why the considered process can be extended bidirectionally. The initial version of the article was drafted by K. Czudek. Subsequently, T. Szarek and I made necessary revisions, amendments, and completed the details of the proof.

5 Description of the results from articles not related to the topic of the scientific achievement

5.1 List of scientific articles

Latest result (paper under review)

[R1] H. Wojewódka-Ściążko, Z. Puchała, K. Korzekwa, *Resource engines*, under review in Quantum, arXiv:2304.09559 [quant-ph] (2023), DOI 10.48550/arXiv.2304.09559.

The results on randomness amplification obtained during the postdoctoral internship at the National Quantum Information Centre in Gdańsk

[Q1] H. Wojewódka, F.G.S.L. Brandão, A. Grudka, K. Horodecki, M. Horodecki, P. Horodecki, M. Pawłowski, R. Ramanathan, M. Stankiewicz, *Amplifying the randomness of weak sources correlated with devices*, IEEE Trans. Inf. Theory. **63** (2017), no. 11, 7592–7611, DOI 10.1109/TIT. 2017.2738010.

[Q2] R. Ramanathan, F.G.S.L. Brandão, K. Horodecki, M. Horodecki, P. Horodecki, H. Wojewódka, *Randomness amplification under minimal fundamental assumptions on the devices*, Phys. Rev. Lett. **117** (2016), no. 23, 230501, DOI 10.1103/PhysRevLett.117. 230501.

[Q3] F.G.S.L. Brandão, R. Ramanathan, A. Grudka, K. Horodecki, M. Horodecki, P. Horodecki, T. Szarek, H. Wojewódka, *Realistic noise-tolerant randomness amplification using finite number of devices*, Nat. Commun. **7** (2016), no. 11345, DOI 10.1038/ncomms11345.

5.2 Description of the results

5.2.1 Introduction

In connection with the implementation of interdisciplinary research projects focused around quantum information theory, that is

- the European grant *Randomness and Quantum Entanglement* (during my internship at the National Quantum Information Centre in Gdańsk in the years 2013–2016)
- and the FNP TEAM-NET grant *Near-term Quantum Computers: Challenges, optimal implementations and applications* (during my internship at the Institute of Theoretical and Applied Informatics of the Polish Academy of Sciences in Gliwice),

my research interests also concern the application of some probabilistic methods in quantum information theory (including random number generation, randomness amplification, quantum cryptography, as well as quantum resource theories or quantum thermodynamics).

In the next section, I will describe my latest results on resource engines [R1] (the concept of resource engines has been recently introduced to the literature by me, prof. dr hab. Zbigniew Puchała and dr hab. Kamil Korzekwa, with whom I am still working on this topic) and on randomness amplification (a series of articles [Q1]–[Q3] was prepared jointly with a group of scientists led by prof. dr hab. Michał Horodecki, with whom I continue to share some future research plans, this time on quantum thermodynamics).

5.2.2 Overview of the results

The results on resource engines

Given how fruitful the thermodynamic inspirations have been so far for quantum resource theories, in [R1] we aim at pushing this analogy one step further. In contrast to current approaches focused on scenarios with one set of constrained free operations (inspired by the access to a single heat bath), we propose to investigate the performance of ***resource engines***, which generalise the concept of heat engines by replacing the access to two heat baths at different temperatures with two arbitrary constraints on state transformations.

More precisely, we consider two agents (traditionally refereed to as Alice and Bob), each of which is facing a different constraint, meaning that each of them can only prepare a subset of free states, F_A and F_B, and can only perform quantum operations from a subset of free operations, $\mathcal{F}_A$ and $\mathcal{F}_B$. Now, the idea is to imitate the action of a two-stroke heat engine: instead of subsequently connecting the system to the hot and cold bath, it is sent to Alice and Bob in turns and they can perform any operation on it from their constrained sets $\mathcal{F}_A$ and $\mathcal{F}_B$. Since the free states and operations of Alice will generally be resourceful with respect to Bob's constraints (and vice versa), a number of such communication rounds with locally constrained operations (i.e., strokes of a resource engine) may generate quantum states outside of F_A and F_B. Thus, by fusing two resource theories described by $(F_A, \mathcal{F}_A)$ and $(F_B, \mathcal{F}_B)$, one can obtain a new resource theory with free operations $\mathcal{F}_{AB} \supseteq \mathcal{F}_A \cup \mathcal{F}_B$ and free states $F_{AB} \supseteq F_A \cup F_B$.

A number of natural questions then arise. First: *Can a resource engine defined by given two constraints generate a full set of quantum operations, or at least approach every element of this set arbitrarily well with the number of strokes going to infinity?*. Alternatively: *Can it achieve all possible final states starting from states belonging to F_A or F_B?*. If the answer to any of these questions is yes, then further questions arise, for instance: *Can we bound the number of strokes needed to generate every operation or state?*, and also: *If there exists a state that is maximally resourceful with respect to both Alice's and Bob's constraints, what is the minimal number of strokes needed to create it?*. Note that given that each stroke takes a fixed amount of time, this effectively corresponds to studying the optimal power of a resource engine. One can also ask about the equivalent of engine's efficiency. Namely, whenever Bob gets a state from Alice and transforms it using an operation from $\mathcal{F}_B$, he necessarily decreases the resource content of the state with respect to his constraint, but may increase it with respect to Alice's constraint. Thus, one may investigate the optimal trade-off, i.e., the efficiency of transforming his resource into Alice's resource.

The main motivation behind introducing and investigating the resource engine was due to the fact that it provides a natural way of fusing two (or more) resource theories, in the spirit of recent works on multi-resource theories [1]. In a sense, it allows one to study how compatible various constraints on allowed transformations are. In [R1] we start this kind of research by introducing ideas and analysing toy examples, but we hope that a formal mathematical framework allowing for fusing arbitrary resource theories can be developed. Two potential ways to achieve this could be to extend to multiple resources the framework of general convex resource theories [2], or to quantify resource-dependent complexity of quantum channels [3].

In Section II in [R1], we analyze a **resource-theoretic perspective on standard heat engines**. Alice and Bob are constrained to having access to heat baths at different temperatures α and β ($\beta < \alpha$). Thus $\mathcal{F}_A$ and $\mathcal{F}_B$ are the sets of thermal operations with inverse temperatures α and β, respectively, and the corresponding sets of free states are $F_A = \{\gamma\}$ and $F_B = \{\mathbf{\Gamma}\}$, where

$$\gamma_k = \frac{e^{-\alpha E_k}}{\sum_{i=1}^d e^{-\alpha E_i}}, \qquad \Gamma_k = \frac{e^{-\beta E_k}}{\sum_{i=1}^d e^{-\beta E_i}},$$

and $\{E_i\}$ denotes the energy levels of the system. We first fully describe an elementary example of a two-level system. Then, given athermality engines with arbitrary d-dimensional systems, **we lower and upper bound the set of achievable states** F_{AB} (see Corollary 2 and Proposition 5 in [R1]). Finally, **we prove that the set F_{AB} of free states arising from fusing two resource theories of thermodynamics, one with finite temperature β and the other with infinite temperature α, is given by the full probability simplex. Moreover, the convergence to the full simplex (with the number of strokes) is exponential**.

Section III is devoted to the concept of **unitary coherence engines**. Alice and Bob are constrained to only performing unitary operations diagonal in their fixed bases, so that coherence with respect to these bases is a resource for them. More formaly, the sets of free states are given by

$$F_A = \{|i\rangle\}_{i=1}^d, \qquad F_B = \{U^\dagger |i\rangle\}_{i=1}^d,$$

where U is a fixed unitary matrix over the field $\mathbb{C}$ describing the relative orientation of the two bases. The sets $\mathcal{F}_A$ and $\mathcal{F}_B$ of free operations are then unitaries diagonal in the distiguished bases. Again, we first investigate the simplest case of a two-level system. Referring to the papers [4, 5, 6] on generating the rotation group, we observe that Alice and Bob can, by means of a resource engine, generate any unitary matrix of order 2, with $\lceil \pi/\alpha \rceil + 1$ strokes, and any pure quantum state, with $\lceil \pi/2\alpha \rceil + 1$ strokes. Then we proceed to a general case of a d-level system and **discuss the conditions under which the full set of unitary operations can be performed jointly by Alice and Bob, i.e., when $\mathcal{F}_{AB}$ becomes the full set of unitary operations**. Namely, we prove (in Theorem 9 in [R1]) that if for U, appearing in the definition of $\mathcal{F}_B$, there exists a constant $M \in (0, \infty)$ such that the corresponding matrix $(P_U^T P_U)^M$, with $P_U = [p_{ij}]_{i,j=1}^d$ given by

$$p_{ij} = \begin{cases} 0 & \text{for } u_{ij} = 0 \\ 1 & \text{for } u_{ij} \neq 0 \end{cases},$$

has all non-zero entries, then any unitary matrix can be written as a product comprised of unitary matrices from $\mathcal{F}_A$ and from $\mathcal{F}_B$. In the proof we refer to the results concerning subgroups of the unitary group that contain the group of diagonal matrices (see [7]). Later, in Propositions 11 and 12, **we analyse the number of strokes N needed to get all these operations, presenting both lower and upper bounds for N**. Finally, we discuss the problem of using the resource engine to produce a state that is simultaneously maximally resourceful for both Alice and Bob (cf. Section III E).

In the end, let us point out that the resource engine perspective may provide a unified framework to study seemingly unrelated problems within the field of quantum information. For instance, our toy example of unitary coherence engines can be directly related to the problem of compiling universal quantum circuits via Hamiltonian control (see, e.g., [8, 9, 10]). Thus, the results on resource engines could be, e.g., applied to optimise quantum control and circuit compilation. Moreover, a similar connection can be made with the problem of performing arbitrary optical linear transformations [11, 12] (e.g., Alice can be restricted to performing only phase masks, while Bob can only perform Fourier transforms or beam splitters). We elaborate more on potential uses of resource engines in Section IV of [R1].

The results on randomness amplification

Randomness is a fundamental concept, with implications from security of modern data systems, to fundamental laws of nature and even the philosophy of science. Randomness is called certified if it describes events that cannot be pre-determined by an external adversary. Traditional random number generators are based on classical physics, which is deterministic. Therefore, the output randomness cannot be trusted without further assumptions.

In view of that, computer scientists have considered the weaker task of ***amplifying imperfect randomness***. Overall, the idea is to use the inputs from a somewhat random, but potentially almost deterministic, source and obtain perfectly random output bits (where by perfectly random we mean uniformly distributed and independent of each other). In classical information theory, randomness amplification from a single weak source is unattainable ([13]). However, it becomes possible, if the no-signaling principle is assumed and quantum-mechanical correlations are used (see, e.g. [14]). Such correlations are revealed operationally through the violation of Bell inequalities.

As a model of a weak source to be amplified, we consider an ***ε-SV source*** (named after M. Santha and U. Vazirani [13]), where ε is a parameter which indicates how far we are from full randomness. An ε-SV source is given by a probability distribution $\mathbb{P}$ over bit strings such that

$$\begin{aligned}&(0.5-\varepsilon) \leq \mathbb{P}(\varphi_1 = x_1|e) \leq (0.5+\varepsilon),\\ &(0.5-\varepsilon) \leq \mathbb{P}(\varphi_{i+1} = x_{i+1}|\varphi_1 = x_1, \ldots, \varphi_i = x_i, e) \leq (0.5+\varepsilon)\end{aligned} \tag{26}$$

for every $i \geq 1$ and all $x_1, \ldots, x_{i+1} \in \{0,1\}$, where e represents an arbitrary random variable prior to φ_1, which can influence the distributions of $\varphi_1, \ldots, \varphi_n, \ldots$. Note that, when $\varepsilon = 0$, bits are fully random, while they can be even fully deterministic when $\varepsilon = 0.5$.

For a long time it was unclear whether randomness amplification (RA) is a realistic task, as the proposals (existing at that moment) either did not tolerate noise or required an unbounded number of different devices.

In [Q3] we provide an error-tolerant protocol using a finite number of devices for amplifying arbitrary weak randomness into nearly perfect random bits, which are secure against a no-signalling adversary. The correctness of the protocol is assessed by violating a Bell inequality, with the degree of violation determining the noise tolerance threshold. In the proofs we combine results from the classical theory of extractors (cf. [15, 16, 17], the recently discovered information-theoretic approach to the de Finetti theorem [18] and the Azuma–Hoeffding inequality [19]. Summary of [Q3].

Our paper [Q3] has been cited 37 times and has inspired several significant results. For example, the authors of [20], drawing heavily on the ideas presented in [Q3] and those in [21], recently developed an end-to-end and practical randomness amplification and privatization protocol based on Bell tests. Remarkably, they demonstrated the protocol on various quantum computers, despite them not being specifically designed for this purpose. This semi-device-independent approach allowed the protocol to generate (near-)perfectly unbiased and private random numbers on today's quantum computers.

Summary of [Q2].

On the other hand, **for reasons of practical relevance, the crucial question is whether cryptographically secure random bits can be produced under the minimal conditions necessary for the task, that is, using only two nonsignaling components and in a situation where the violation of a Bell inequality only guarantees that some outcomes of the device for specific inputs exhibit randomness. The question is yes and it is answered in** [Q2]. More precisely, we present a device-independent protocol for RA of SV sources using a device consisting of two nonsignaling components. We show that the protocol can amplify any such source that is not fully deterministic into a fully random source while tolerating a constant noise rate and prove the composable security of the protocol against general no-signaling adversaries. Our main innovation is the proof that even the partial randomness certified by the two-party Bell test (a single input-output pair $(\mathbf{u}^*, \mathbf{x}^*)$ for which the conditional probability $\mathbb{P}(\mathbf{x}^*|\mathbf{u}^*)$ is bounded away from 1 for all no-signaling strategies that optimally violate the Bell inequality) can be used for amplification. We introduce the methodology of a partial tomographic procedure on the empirical statistics obtained in the Bell test that ensures that the outputs constitute a linear min-entropy source of randomness. As a technical novelty that may be of independent interest, we prove that the SV source satisfies an exponential concentration property given by a recently discovered generalized Chernoff bound.

Summary of [Q1].

In [Q1] we focus on the possibility of enhancing the randomness of sources by means of devices dependent on them (ensuring complete independence of the device from the source is difficult to implement in practice). Instead of requiring a source and a device to be independent, we only limit the correlations between them by one constraint, which we call the ***SV-condition for boxes*** (for details see Section IV B in [Q1]). **We prove the protocol's security (see Section VII and Figure 6 in** [Q1]**, where the protocol is presented) against a certain class of attacks using correlations between the source and the device (in the context of a finite device framework against a no-signaling adversary).** More precisely, we prove explicitly that the most malicious correlations (between a source and a device) are not allowed due to the assumption that an ε-SV source remains an ε-SV source even upon obtaining the inputs and outputs from boxes. Hence, randomness amplification is still possible. Our new method of proof allows to analyze an attack where an adversary sends to the honest parties those boxes that are particularly adapted to their measurement settings, as well as to the hashing function applied. We explain the dangers of such attacks with an explicit example in Section III of [Q1].

Let us here indicate that other researchers have also attempted to relax the independence assumption. The authors of [22] approach the problem using a quantum formalism, while those in [23], which was announced after the initial version of ([Q1]), demonstrate (in the same spirit) security against no-signaling adversaries, albeit with a greater number of devices. Our approach is different and independent of those proposed in [22] and [23]. We believe that the results presented in our paper [Q1] offer a novel perspective on randomness amplification and, due to the transparency of the assumptions, are also significant in the broader context of obtaining secure cryptographic key bits.

References

[1] C. Sparaciari, L. Del Rio, C. M. Scandolo, P. Faist, J. Oppenheim, *The first law of general quantum resource theories*, Quantum **4** (2020), no. 259.

[2] R. Takagi, B. Regula, *General resource theories in quantum mechanics and beyond: operational characterization via discrimination tasks*, Phys. Rev. X **9** (2019), no. 031053.

[3] R. Araiza, Y. Chen, M. Junge, and P. Wu, *Resource-dependent complexity of quantum channels*, arXiv:2303.11304 [quant-ph] (2023).

[4] F. Lowenthal, *Uniform finite generation of the rotation group*, Rocky Mt. J. Math. **1** (1971), no. 575.

[5] F. Lowenthal, *Uniform finite generation of SU(2) and SL(2, R)*, Canad. J. Math. **24** (1972), no. 713.

[6] M. Hamada, *The minimum number of rotations about two axes for constructing an arbitrarily fixed rotation*, R. Soc. Open Sci.**1** (2014), no. 3.

[7] Z. Borevich and S. Krupetskij, *Subgroups of the unitary group that contain the group of diagonal matrices*, J. Sov. Math. **17** (1981), 1951–1959.

[8] S. Lloyd, *Almost any quantum logic gate is universal*, Phys. Rev. Lett. **75** (1995), 346.

[9] N. Weaver, *On the universality of almost every quantum logic gate*, J. Math. Phys. **41** (2000), no. 240.

[10] V. Ramakrishna, K. L. Flores, H. Rabitz, R. J. Ober, *Quantum control by decompositions of SU(2)*, Phys. Rev. A **62** (2000), 053409.

[11] V.L. Pastor, J. Lundeen, F. Marquardt, *Arbitrary optical wave evolution with Fourier transforms and phase masks*, Opt. Express **29** (2021), no. 23, 38441–38450.

[12] L. Pereira, A. Rojas, G. Cañas, G. Lima, A. Delgado, A. Cabello, *Minimum optical depth multi-port interferometers for approximating any unitary transformation and any pure state*, arXiv:2303.11304 [quant-ph] (2023).

[13] M. Santha, U.V. Vazirani, *Generating quasi-random sequences from slightly-random sources*, Proc. 25th IEEE Symp. Found. Comput. Sci. (FOCS'84) (1984), 434–440.

[14] R. Colbeck, R. Renner, *Free randomness can be amplified* Nat. Phys. **8** (2012), 450–453.

[15] E. Chattopadhyay, D. Zuckerman, *Explicit two-source extractors and resilient functions*, Electronic colloquium on computational complexity, Revision 1 of Report No 119 (2015).

[16] B. Chor, O. Goldreich, *Unbiased bits from sources of weak randomness and probabilistic communication complexity*, IEEE 26th Annu. Symp. Found. Comput. Sci. (1985), 429–442.

[17] X. Li, *Extractors for a constant number of independent sources with polylogarithmic min-entropy*, IEEE 54th Annu. Symp. Found. Comput. Sci. (FOCS) (2013), 100–109.

[18] F.G.S.L. Brandão, A.W. Harrow, *Quantum de Finetti theorems under local measurements with applications*, Proc. 45th Annu. ACM Symp. Theory Comput. (STOC'13) (2013), 861–870.

[19] S. Pironio, S. Massar, *Security of practical private randomness generation*, Phys. Rev. A **87** (2013), no. 012336.

[20] C. Foreman, S. Wright, A. Edgington, M. Berta, F.J. Curchod, *Practical randomness amplification and privatisation with implementations on quantum computers*, Quantum **7** (2023), no. 969.

[21] M. Kessler, R. Arnon-Friedman, *Device-independent randomness amplification and privatization*, IEEE Journal on Selected Areas in Information Theory **1** (2020), no. 2, 568–584.

[22] K.-M. Chung, Y. Shi, X. Wu, *Physical randomness extractors: generating random numbers with minimal assumptions*, arXiv:1402.4797 [quant-ph] (2014).

[23] K.-M. Chung, Y. Shi, X. Wu, *General randomness amplification with non-signaling security*, Available at https://api.semanticscholar.org/CorpusID:17461712 (2016).

6 Presentation of significant scientific activity carried out at more than one university or scientific institution, especially at foreign institutions

Institute of Mathematics, University of Silesia in Katowice (permanent position)

In October 2016, I started working at the **Institute of Mathematics of the University of Silesia in Katowice** (IM UŚ). I am still an assistant professor there, conducting research on asymptotics and ergodic properties of certain Markov dynamical systems (see, e.g., [H2]–[H5] or [E2]–[E4]).

Invitations to present talks and lectures at various conferences, both nationally and internationally, including

- the international conference ***Mathematical Modeling with Measures: Where Applications, Probability and Determinism Meet*** (Leiden, 3–7 December 2018; **invited talk**: *Useful versions of limit theorems for certain non-stationary Markov chains*),
- ***The 17th Conference on Probability Theory*** (*XVII Konferencja z Probabilistyki*, Będlewo, 22–26 May 2023; **plenary lecture**: *Central limit theorem for Markov processes which are exponentially ergodic in the Fortet–Mourier norm*),

as well as during the seminars of the Institute of Mathematics, Polish Academy of Sciences (IMPAN) in Warsaw (15 April 2019 – the seminar on *Differential Equations*; 28 March 2023 – the seminar on *Stochastic Processess*), serve as an acknowledgment of strong recognition of my scientific contributions. Overall, I have given talks at 10 international mathematical conferences, including the ***Bernoulli-IMS 10th World Congress in Probability and Statistics*** (Seoul, 19–23 July 2021).

From 2 March 2020 to 31 May 2021, I had a **break in my scientific activity**, initially due to incapacity for work, followed by maternity and parental leaves.

From 1 October 2021 to 30 June 2023, I was on an unpaid leave at the University of Silesia in Katowice (at that time I was involved in the implementation of research projects at the Institute of Theoretical and Applied Informatics of the Polish Academy of Sciences in Gliwice).

Institute of Theoretical and Applied Informatics, Polish Academy of Sciences (postdoctoral fellowship)

In September 2019, I was invited by prof. dr hab. Zbigniew Puchała, the leader of the *Quantum Machine Learning* group in the project ***Near-term Quantum Computers: Challenges, optimal implementations and applications*** (grant number: POIR.04.04.00-00-17C1/18-00, the TEAM-NET grant funded by the Foundation for Polish Science), to collaborate on this project. In December 2019, I took on additional employment (half-time position) at the **Institute of Theoretical and Applied Informatics of the Polish Academy of Sciences** (IITiS PAN) in Gliwice.

In the period from 1 October 2021 to 30 June 2023, when I was on an unpaid leave at the University of Silesia in Katowice, my working time in the TEAM-NET project was increased to full-time. Additionally, I was employed in the project of dr hab. Bartłomiej Gardas, entitled *Simulations of physical systems with near-term annealing technology* (grant number: 2020/38/E/ST3/00269, Sonata BIS 10 grant funded by the National Science Center).

During that time, I became particularly interested in the theory of quantum resources and established an intensive scientific cooperation with dr hab. Kamil Korzekwa from the **Faculty of Physics, Astronomy and Applied Computer Science at the Jagiellonian University in Kraków**. The effect of this collaboration is the article [R1], which I have already promoted at the following international conferences:

- *Quantum Resources: from Mathematical Foundations to Operational Characterization* (Singapore, 5–8 December 2022),
- *Near Term Quantum Computing 2020(+3)* (Warsaw, 22–24 March 2023)
- and *14th KCIK-ICTQT Symposium on Quantum Information* (Sopot, 18–20 May 2023, **invited talk**).

Our ideas were enthusiastically received by the participants of these conferences.

Mathematical Institute, Leiden University (long-term international cooperation and internships)

From 9 December to 14 December 2013, and from 15 December to 19 December 2014, I had my first study visits to the **Mathematical Institute at Leiden University** (MI LU), thanks to which I established an intensive scientific cooperation with Dr. Sander C. Hille. Our joint research resulted in the publication of the articles [D1] and [D2], as well as subsequent invitations to MI LU.

During the spans of 18 June – 18 July 2017 and 30 January – 1 March 2018, I completed two **month-long internships**, the latter of which was financed by the *Outgoing Scholarship*, awarded to me under the START FNP 2017 program. Further, shorter visits to MI LU took place during 2–18 December 2018 and 3–12 February 2019, as part of the implementation of the Miniatura NCN project and *Small Grant* from the Rector of the University of Silesia in Katowice (*Mały Grant* Rektora Uniwersytetu Śląskiego w Katowicach), respectively.

The scientific cooperation with Dr. Sander C. Hille in the years 2017–2019 has resulted in the publications [H3] and [E4], in which we establish certain properties of invariant measures and the law of the iterated logarithm for the considered class of piecewise deterministic Markov processes, respectively.

Finally, from 31 August to 30 November 2022, I undertook a **three-month foreign internship** at MI LU, during which, together with Dr. Sander C. Hille, we initiated an entirely new project focusing on convergence rates to stationary distributions in certain Monte Carlo simulations. We also extended the collaboration to include Dr. Joris Bierkens from the **Delft University of Technology**, whom I had the opportunity to meet at the mini-symposia *Markov operators and dynamical systems in spaces of measures* organized during my stay in Leiden. These research efforts continue today.

Institute of Theoretical Physics and Astrophysics, University of Gdańsk; National Quantum Information Centre in Gdańsk (postdoctoral fellowship)

In October 2013, while still a PhD student, I started working as an assistant (researcher) in the **Institute of Theoretical Physics and Astrophysics, University of Gdańsk** (IFTiA UG), with a secondment to work at the **National Quantum Information Centre in Gdańsk** (KCIK), where, under the supervision of prof. dr hab. Michał Horodecki, I implemented the **European grant *Randomness and Quantum Entanglement*** (acronym RAQUEL, project number 323970, the Seventh Framework Program of the European Union). After completing doctoral studies and obtaining a doctoral degree (in July 2015), I continued my employment at IFTiA UG (and KCIK), **as a postdoctoral research fellow**, until the end of September 2016. During this period, I dealt with the problem of randomness generation, which is important, among others, from a cryptography perspective. The outcome of my research consists of three publications [Q1]-[Q3].

Participation in the project *Randomness and Quantum Entanglement* gave me the opportunity to work within an international and interdisciplinary team consisting of physicists, mathematicians, and computer scientists. Between the years 2013 and 2015, the project contributors, i.e. employees of the Masaryk University in Brno, the Swiss Federal Institute of Technology in Zürich, the Free University of Berlin, the Autonomous University of Barcelona, the University of Latvia and the University of Gdańsk, met every year at conferences ***RAQUEL Scientific Meeting*, hosted successively in Brno, Barcelona and Sopot** (with the final conference co-organized by me and dr Piotr Ćwikliński). During these meetings, we presented our results, engaged in discussions on open research issues, and fostered productive exchanges within the team.

During that period, I also had the opportunity to undertake several scientific visits, including visits to the **Institute of Physics at Adam Mickiewicz University in Poznań** (on 12–16 May 2014 and from 8–11 September 2014 – in cooperation with prof. dr hab. Andrzej Grudka, which resulted in the publications [Q1] and [Q3]) or to the **Institute of Theoretical Physics of the Swiss Federal University of Technology in Zürich** (on 6–12 December 2015 – for scientific consultation with Prof. Dr. Renato Renner and his research group, regarding the work [Q1]). Moreover, I participated in numerous international conferences, workshops, and seminars, including those held in Sydney, Barcelona, Seefeld, Washington, and Berlin, where I presented and promoted the results of my research.

While preparing the publications [Q1]-[Q3], I established scientific cooperation with Prof. Fernando G.S.L. Brandão from the **California Institute of Technology** (Caltech) (formerly from the *Miscrosoft* and the University College London).

At the end of the postdoctoral fellowship at KCIK in Gdańsk, I was honored with an invitation to deliver a presentation entitled *Towards realistic randomness amplification* during the *Symposium KCIK* (Sopot, 22–24 May 2016). Additionally, I was **invited by the program committee of the *44th Polish Physics Congress*** (Wrocław, 10–15 September 2017) to present the results from the series of works [Q1]–[Q3]. I was also invited to give lectures at the seminar *Chaos and Quantum Information*, organized by the Quantum Optics Division of the Jagiellonian University in Kraków (Kraków, 13 November 2017), as well during the conference *Quantum Foundations and Beyond* (Sopot, 8–9 December 2017).

In 2017, I received a **foreign scholarship from the Henri Poincaré Institute (IHP) in Paris**, which enabled me to participate in the program *T3-2017 Analysis in Quantum Information Theory* held at the IHP in Paris (this included a **month-long research visit**, from 19 November to 17 December 2017, during which I presented my results at the *Main Conference Quantum Information Theory, IHP Trimester*).

Institute of Mathematics, University of Gdańsk (doctoral studies)

In October 2011, I started the *Environmental Doctoral Studies in Mathematics and Computer Science* at the **Institute of Mathematics, University of Gdańsk**, and I completed them in July 2015. Throughout this period, my interests were focused primarily on the analysis of some random dynamical system, used, among others, to describe the process of cell division (see [D1]–[D3]). I promoted my results by giving talks at the conferences, including, international conferences such as the *CNRS-PAN Mathematics Summer Institute* in Kraków in 2013 and 2015.

7 Presentation of teaching and organizational achievements, as well as achievements in popularization of science

Teaching achievements

Conducting lectures and exercises

- In the academic year 2021/2022, I conducted an online lecture entitled *Probability and Statistics* (in English) for students of the **Master's program in *Quantum Information Technology*, run by the University of Gdańsk**.
- From October 2016 to February 2020, I conducted numerous **lectures and classes at the Faculty of Science and Technology of the University of Silesia in Katowice** (according to my annual teaching load of 210 hours), including courses such as *Introduction to Probability Theory*, *Probability Theory*, *Elements of Statistics*, *Fundamentals of Probabilistic Methods and Statistics*, *Introduction to Computer Science*, as well as *Problem Workshops*, *Diploma Seminars*, and *Master's Laboratory.*
- In the academic year 2015/2016, I conducted lectures on *Stochastic Differential Equations* and *Applications of Semi-Markov Processes* at the **Department of Probability Theory and Biomathematics, Faculty of Technical Physics and Applied Mathematics, Gdańsk University of Technology**.

Supervising students

- I was the **supervisor of two master's theses** defended in 2019 (*Base Norm in the Problem of Quantum Measurements Discrimination* by Paulina Lewandowska, and *Some Asymptotic Properties of Markov Operators* by Ryszard Kukulski, which received **distinction**).
- I was the **supervisor of seven bachelor's theses**, defended in 2019 (by Paulina Gamrat, Małgorzata Kubicka and Natalia Czerniszew) and in 2017 (by Agata Dytkowska, Katarzyna Chmura, Ryszard Kukulski and Agata Malisz).

Organizational achievements

- I co-organized (together with prof. dr hab. Katarzyna Horbacz and dr Dawid Czapla) the first two editions of the ***Microconference on Stochastic Processes*** (Katowice, 6–7 June 2023 and 20–22 June 2022).
- I was a member of the **Scientific Council of the Institute of Mathematics** of the University of Silesia in Katowice in the academic years 2020/2021 and 2021/2022.
- I was a member of the **Teaching Council for Mathematics** at the Faculty of Science and Technology of the University of Silesia in Katowice in the academic year 2019/2020.
- In the years 2015 and 2016, I was co-running (together with dr Paweł Mazurek) the **website of the National Quantum Information Centre in Gdańsk**.
- I co-organized (together with dr Piotr Ćwikliński) the international conference ***3rd RAQUEL Scientific Meeting*** (Sopot, 8–9 October 2015).

Achievements in popularization of science

- I prepared and conducted the ***XXXV and XXXVI National Congress of Mathematicians*** (*Ogólnopolski Sejmik Matematyków*) (Szczyrk, 6–9 June 2019 and 14–17 June 2018).
- I gave a talk entitled *What is randomness and can it be amplified?* at the ***XII Pi Day Celebration***, an outreach event organized annually by the Faculty of Science and Technology of the University of Silesia in Katowice (Katowice, 14 March 2018).
- I conducted workshops for high school and junior high school students as part of the program ***Talented from Pomerania, Academic Meetings*** organized by the Foundation for the Development of the University of Gdańsk and the University of Gdańsk in the years 2012 and 2013.
- I won the **1st prize in the *Mathematical Tale Competition***, organized by the Institute of Mathematics of the University of Gdańsk and the Gdańsk Community Foundation in 2011, for a fairy tale entitled *Friends from Gdańsk*.
- I hosted the festival event *What mathematics has to do with love?* during the ***8th Baltic Science Festival*** (Gdańsk–Sopot, 27 May 2010).

8 Other information regarding professional career

Selected scholarships and awards

2018

Individual Award of the 3rd degree for Research Activities, conferred by the Rector of the University of Silesia in Katowice.

2017

START Stipend awarded by the Foundation for Polish Science for outstanding young researchers under the age of thirty, representing all fields of science.

The special **Prof. Barbara Skarga Stipend** awarded by the Foundation for Polish Science to the winner of the START programme whose research (quoting from the official website of the FNP) proves boldest in breaking down barriers between academic disciplines, opening new research perspectives, and creating new values in the field of science.

START Outgoing Scholarship awarded by the Foundation for Polish Science to selected winners of the START program, enabling a four-week scientific internship at a research center of their choice (in my case, the *Mathematical Institute at the Leiden University* in the Netherlands).

Foreign scholarship from the Henri Poincaré Institute in Paris, enabling participation in the program *T3-2017 Analysis in Quantum Information Theory* (*IHP Trimester*).

Management of research projects

2018–2019

Mathematical modeling with measures
(**MINIATURA** grant from the National Science Center Poland under the project number 2018/02/X/ST1/01518).

Properties and applications of certain random Markov dynamical systems
(**SMALL GRANT**, funded by the Rector of the University of Silesia in Katowice, as part of a program supporting grant initiatives and enhancing opportunities for employees of the University of Silesia in Katowice to acquire projects in external competitions).