Attachment No 3 to the Application for Entering the Habilitation Procedure - summary of professional accomplishments

1. Name: Włodzimierz Fechner.
2. Academic degrees:
    1. Master of Mathematics: University of Silesia, Institute of Mathematics, June, 1st. 2003.
    The title of Master Thesis: *Functional Equations in Rätz Space* (in Polish),
    Supervisor: Prof. dr hab. Roman Ger.
    2. PhD in Mathematics: University of Silesia, Institute of Mathematics, July 2nd 2007.
    The Title of Dissertation: *Functional Inequalities Connected with Quadratic Functionals* (in Polish),
    Supervisor: Prof. dr hab. Roman Ger.
3. Information about the employment in scientific institutions:
    1. March 15, 2007 - June 30, 2007: research-and-teaching assistant (part-time employment), University of Silesia, Institute of Mathematics.
    2. October 1st, 2007 - now: assistant professor (full post), University of Silesia, Institute of Mathematics.
4. The scientific achievement spoken of in a respective act of Polish law.
    (a) The monographic set of publications entitled *Functional Inequalities in several variables*.
    (b) The list publications of the monographic set of publications:
        [F1] Włodzimierz Fechner, *Functional characterization of a sharpening of the triangle inequality*, Math. Inequal. Appl. 13/3 (2010), 571–578.
        [F2] Włodzimierz Fechner, *On some composite functional inequalities*, Aequationes Math. 79/3 (2010), 307–314.

[F3] Włodzimierz Fechner, *Four inequalities of Volkmann type*, J. Math. Inequal. 5/4 (2011), 463–472.

[F4] Włodzimierz Fechner, *A note on alienation for functional inequalities*, J. Math. Anal. Appl. 385 (2012), 202–207.

[F5] Włodzimierz Fechner, *Hlawka's functional inequality*, Aequationes Math. (2012) doi=10.1007/s00010-012-0178-2.

[F6] Włodzimierz Fechner, *Inequalities connected with averaging operators*, Indagationes Math. 24 (2013), 305–312.

[F7] Włodzimierz Fechner, *Functional inequalities motivated by the Lax-Milgram theorem*, J. Math. Anal. Appl. 402 (2013), 411–414.

(c) The description of the scientific goal of the foregoing papers and results obtained, jointly with their potential applications:

The aim of the monographic set of publications is to examine some problems of functional inequalities in several variables, its applications and connections with other mathematical disciplines. Solutions of the posed problems are contributions of the habilitation candidate to the development of the theory of functional inequalities. Tools and the proof techniques applied are far from standard methods used for solving similar problems and are additional contributions of the habilitation candidate to the development of this theory. Moreover, this set of publications reveals new connections between the theory of functional inequalities and elements of operator theory and of multifunction theory.

Introduction.

We begin with a short description of two most basic functional inequalities being a building blocks in our research, namely the inequality defining Jensen convex functions and the inequality defining subadditive functions. The latter one plays the crucial role in our further research.

Assume that $(X, +)$ is an Abelian semigroup with unique division by two, $D \subset X$ is a set such that $\frac{1}{2}(x+y) \in D$ for all $x, y \in D$ and $f\colon D \to \mathbb{R}$ is an arbitrary function. We say that f is *Jensen convex*, if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \quad x, y \in D.$$

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Assume additionally that X is a real linear space and the set D is convex. A function $f\colon D \to \mathbb{R}$ is *convex*, if

$$f\left(\lambda x + (1-\lambda)y\right) \leq \lambda f(x) + (1-\lambda) f(y), \quad x, y \in D,\ \lambda \in [0,1].$$

Every continuous Jensen convex function defined on a convex subset of a real linear-topological space is convex. Since there exist discontinuous linear functionals, there exist also discontinuous convex functions. A discontinuous additive function on the real line is an example of a Jensen convex function which is not convex. On the other hand, slight regularity conditions upon a Jensen convex function imply its continuity. The Bernstein-Doetsch theorem says that every Jensen convex function defined on an open and convex subset of a real linear topological space which is bounded from above on some non-empty open set is convex. On the other hand, Sierpiński's theorem states that each Lebesgue measurable and Jensen convex function defined on an open and convex subset of $\mathbb{R}^n$ is continuous. More details concerning these notions can be found in the monograph of Marek Kuczma [12].

The situation is much more difficult in the case of the second functional inequality we are interested in. Assume that $(X, +)$ is an Abelian semigroup and $f\colon X \to \mathbb{R}$ is an arbitrary function. The function f is *subadditive*, if

$$f\left(x + y\right) \leq f(x) + f(y), \quad x, y \in X.$$

We need to mention that counterparts of Bernstein-Doetsch and Sierpiński theorems are not valid for subadditive functions. In fact, there exist discontinuous subadditive functions with a relative high regularity. A more detailed discussion for this class of functions can be found e.g. in the monograph E. Hille, R.S. Philips [9].

An important class for us are sublinear functions. Let $(X, +)$ be an Abelian semigroup and let $f\colon X \to \mathbb{R}$ be an arbitrary function. The function f is *sublinear*, if it is subadditive and

$$f(2x) = 2f(x), \quad x \in X.$$

In particular, each sublinear function is Jensen convex. Moreover, the following representation of sublinear functions holds true.

Theorem 1 (R. Ger [6]). *Assume that $(X, +)$ is an Abelian group and $f\colon X \to \mathbb{R}$ is an even sublinear function. Then there exist a Banach space E and an additive mapping $A\colon X \to E$ such that f has the following representation:*

$$f(x) = \|A(x)\|, \quad x \in X.$$

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The Banach space E spoken of in the above theorem can be defined explicitly as a space of bounded sequences on some set with the supremum norm.

In the set of papers [F1-F7] we concentrate on selected problems of the theory of functional inequalities, where we develop new proof techniques. We begin with some relatively elementary problems where we apply some tools known earlier. During our investigations more complicated problems occur and an application of classical methods is no longer possible. Thus, in order to tackle them, we are forced to look for nonstandard methods. The first method introduced by the author can be intuitively called "differentiating" an inequality side-by-side or "subtracting" two inequalities side-by-side. This approach allows the author to study new types of problems, so called *composite functional inequalities*. The second tool was introduced in the paper [F7] and is is based on applications of multifunctions and selection theorems for functional inequalities. More details will be given in the last paragraph of the present summary.

Rădulescu's theorem on a characterization of linear-multiplicative operators and a result of Hammer. Alienation of functional inequalities.

Assume that X is a compact Hausdorff topological space and $C(X)$ is the space of all real continuous functions defined on X with the supremum norm. Marius Rădulescu in the paper [16] showed that if an operator $T\colon C(X) \to C(X)$ satisfies the following system of inequalities:

$$\begin{cases} T(f+g) \geq T(f)+T(g), \\ T(f\cdot g) \geq T(f)\cdot T(g), \end{cases} \tag{1}$$

for all $f, g \in C(X)$, then there exist a closed-open set $B \subseteq X$ and a continuous function $\varphi\colon X \to X$ such that

$$T(f) = \chi_B \cdot f \circ \varphi,$$

where χ denotes the characteristic function of a set. In particular, the operator T is linear, multiplicative and continuous.

Jean Dhombres in the paper [4] investigated the following system of equations:

$$\begin{cases} f(x+y) = f(x)+f(y), \\ f(xy) = f(x)f(y), \end{cases} \tag{2}$$

together with the single equations obtained by adding side-by-side equations in the system (2), i.e. the equation:

$$f(x+y)+f(xy) = f(x)+f(y)+f(x)f(y) \tag{3}$$

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and also a more general equation:

$$af(xy) + bf(x)f(y) + cf(x+y) + d(f(x) + f(y)) = 0.$$

Assume that we are given an abstract system of functional equations or inequalities (U) and, respectively, an equation or inequality (E) which is obtained by adding side-by-side both relationships of (U). Of course, every solution of (U) satisfies (E). If the converse implication is true, we say that the *alienation effect* holds for the system (U).

Dhombres in his paper has given conditions under which every solution of equation (3) satisfies the system (2), thus the alienation effect holds for additive and multiplicative Cauchy equations from the system (2). Now we come to the question if a similar behaviour is characteristic for the system of inequalities (1). Intuitively, it would mean that under certain assumption it is possible to "subtract side-by-side" both inequalities of (1). The first result in this direction is due to Claus Hammer in [8]. He has shown that a continuous and differentiable at zero function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the inequality

$$f(x+y) + f(xy) \geq f(x) + f(y) + f(x)f(y), \quad x, y \in \mathbb{R} \tag{4}$$

if and only if f is constant and equal to zero or

$$f(x) = x + \frac{a-1}{a}(e^{ax} - 1), \quad x \in \mathbb{R},$$

where $a = f'(0) \geq 1$.

The aim of the paper [F4] is to continue and extend the investigations of the above problem. We introduce the method, intuitively called by us "differentiating an inequality side-by-side". This is a modification of the method from the mentioned earlier paper of Hammer [8] and will be also employed later in sections describing results from papers [F2] and [F5]. A general idea is based on an inspection of suitable difference quotients and showing by means of the investigated inequality that these quotients fulfil certain estimates. In some cases it is possible to prove that every solution of the investigated inequality has to be continuously differentiable (results of [F4] and [F5]), but in other cases we have to assume a higher regularity of an unknown function (results of [F2]). Further, passing to the limit we usually obtain some (ordinary) differential equation or an inequality and after solving it we derive the general form of solutions of the original problem. In the paper [F4] (published in 2012 but submitted for publication in September 2009) we managed to apply this method in an elementary form. Problems discussed in (written later) papers [F2] and [F5] have been much more difficult, also technically and required significant modifications of our original approach.

The first lemma deals with a general situation with two unknown functions.

Lemma 1 ([F4, Lemma 1]). *Assume that functions $f\colon \mathbb{R}\to\mathbb{R}$ and $g\colon \mathbb{R}\to\mathbb{R}$ are differentiable at zero, g is continuous and $\alpha,\beta\in\mathbb{R}$ are arbitrary constants. If f and g with $f(0)=g(0)=0$ satisfy the following functional inequality*

$$\alpha[f(x+y)-f(x)-f(y)]+\beta[g(xy)-g(x)g(y)]\geq 0, \quad x,y\in\mathbb{R}, \tag{5}$$

then the equality

$$\alpha(f'(x)-f'(0))=\beta g'(0)(g(x)-x), \quad x\in\mathbb{R}.$$

holds. If additionally $\alpha\neq 0$ then f is continuously differentiable.

Observe that this lemma is a generalization of a well-known fact that a subadditive function $f\colon \mathbb{R}\to\mathbb{R}$, which is differentiable at zero and $f(0)=0$ is of the form $f(x)=f'(0)x$ for all $x\in\mathbb{R}$ (it is enough to put $g=0$ and $\alpha=-1$ in the lemma).

Note that Lemma 1 allows us to reduce the problem of solving the functional inequality (5) with two unknown functions to the case with only one unknown function.

Next, we obtain a generalization of the above-mentioned result of Hammer.

Theorem 2 ([F4, Theorem 1]). *Assume that a function $f\colon \mathbb{R}\to\mathbb{R}$ is differentiable at zero and continuous with $f(0)=0$ and that $b,c\in\mathbb{R}$ are nonzero constants. Then f satisfies the inequality*

$$f(x+y)+bf(xy)\geq f(x)+f(y)+cf(x)f(y), \quad x,y\in\mathbb{R} \tag{6}$$

if and only if either $f=0$ or

$$f(x)=\frac{ac-b}{ac^2}[e^{acx}-1]+\frac{b}{c}x, \quad x\in\mathbb{R},$$

where $a=f'(0)$ with $ac>0$ and $(ac-b)bc\geq 0$.

As a corollary we derive that the alienation effect for the investigated system of inequalities holds if and only if either $f=0$ or $f'(0)=\frac{b}{c}$ ([F4, Corollary 1]).

Cases $c=0$ or $b=0$ in the inequality (6) are discussed in the two next theorems.

Theorem 3 ([F4, Theorem 2]). *Assume that a function $f\colon \mathbb{R}\to\mathbb{R}$ is differentiable at zero and continuous with $f(0)=0$ and $b\in\mathbb{R}$ is an arbitrary constant. The function f satisfies the inequality*

$$f(x+y)+bf(xy)\geq f(x)+f(y), \quad x,y\in\mathbb{R}$$

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if and only if

$$f(x) = -\frac{1}{2}abx^2 + ax \quad x \in \mathbb{R},$$

with $a = f'(0)$. Moreover, if additionally $b \neq 0$, then $a \leq 0$.

Theorem 4 ([F4, Theorem 3]). *Assume that a function $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at zero and continuous with $f(0) = 0$ and $c \in \mathbb{R}$ is an arbitrary nonzero constant. A function f satisfies the inequality*

$$f(x+y) \geq f(x) + f(y) + cf(x)f(y), \quad x, y \in \mathbb{R}$$

if and only if

$$f(x) = \frac{1}{c}[e^{acx} - 1], \quad x \in \mathbb{R},$$

with $a = f'(0)$.

Tarski's identity and some strengthening of the triangle condition.

The following elementary identity was observed by Alfred Tarski (see [18]):

$$\big|\, |x| - |y| \,\big| = |x+y| + |x-y| - |x| - |y|, \quad x, y \in \mathbb{R}. \tag{7}$$

Lech Maligranda in [14] shown that (7) is not satisfied in dimensions grater than one. However, in an arbitrary normed linear space the following estimate:

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x+y\| + \|x-y\| - \|x\| - \|y\| \leq \min\{\|x+y\|, \|x-y\|\}. \tag{8}$$

holds. Without using any (!) other properties of the norm we can easily derive from (8) both triangle inequalities: $\|x+y\| \leq \|x\| + \|y\|$ and $\big|\, \|x\| - \|y\| \,\big| \leq \|x-y\|$. Inequality (8), treated as a strengthening of the triangle inequality, is, together with some related relations, a motivation for research published in the papers [F1], [F2] and [F3]. We deal with functional inequalities which, according to its motivations, can be viewed as strengthenings of the subadditivity condition.

The first problem investigated which is connected with the Tarski identity (7) and inequality (8) shown by Maligranda is the functional equation

$$|f(x) - f(y)| = f(x+y) + f(x-y) - f(x) - f(y) \tag{9}$$

and the inequality

$$|f(x) - f(y)| \leq f(x+y) + f(x-y) - f(x) - f(y) \leq \min\{f(x+y), f(x-y)\}. \tag{10}$$

In the paper [F1] we have proved the following theorems.

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Theorem 5 ([F1, Theorem 1]). *Assume that $(X,+)$ is an Abelian group and $f\colon X \to \mathbb{R}$ is a function vanishing at zero. Then f satisfies (10) if and only if there exist a normed linear space $(E, \|\cdot\|)$ and an additive mapping $A\colon X \to E$ such that*

$$f(x) = \|A(x)\|, \quad x \in X.$$

Theorem 6 ([F1, Theorem 2]). *Assume that $(X,+)$ is an Abelian group and $f\colon X \to \mathbb{R}$ is an arbitrary function. Then f satisfies (9) if and only if there exist an additive mapping $A\colon X \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that*

$$f(x) = |A(x)| + c, \quad x \in X.$$

We proved also the Hyers-Ulam stability of (9) ([F1, Theorem 3 and Corollary 4]). In the proof of Theorem 5 we have used Theorem 1 of Ger. On the other hand, Theorem 6 is an easy consequence of Theorem 5.

Peter Volkmann in a personal discussion with the author observed that Theorem 6 can be derived from the results of Chaljub-Simon and Volkmann from the paper [3]. They have investigated the following functional equations

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y), \tag{11}$$
$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)|, \tag{12}$$

for an unknown function $f\colon \mathbb{R} \to \mathbb{R}$.

In view of the above results for function satisfying (10) and its connections with (11) and (12) a natural question arises about the solutions of functional inequalities coming directly from equations (11) and (12):

$$\max\{f(x+y), f(x-y)\} \le f(x) + f(y), \tag{13}$$
$$\max\{f(x+y), f(x-y)\} \ge f(x) + f(y), \tag{14}$$
$$\min\{f(x+y), f(x-y)\} \ge |f(x) - f(y)|, \tag{15}$$
$$\min\{f(x+y), f(x-y)\} \le |f(x) - f(y)|. \tag{16}$$

The above inequalities for functions defined on an Abelian group, called *Volkmann-type Inequalities*, are investigated in the paper [F3]. We observed that there is no symmetry between the respective pairs of inequalities: (13) and (14) and between (15) and (16). Moreover, we proved that every solution of (13) vanishing at zero is an even subadditive function. Additionally, if a solution of this inequality vanishes at some point, then it vanishes at zero. Moreover, the set of zeros of this function forms an additive subgroup of the domain. An analogical behaviour is characteristic for the inequality (15). The difference is connected with the set of zeros: each point from the set of zeros of solutions (15) is a period of f. For inequalities (14) and (16) we provide some examples which show how weak they are in comparison to the converse inequalities.

The technique of "differentiating inequalities side-by-side", already introduced in the previous part, has been applied in the paper [F2] for the following three functional inequalities, which are motivated by (7) and (8):

$$f(f(x)-f(y)) \leq f(x+y)+f(f(x-y))-f(x)-f(y), \quad x,y \in \mathbb{R}, \tag{17}$$
$$f(f(x)-f(y)) \leq f(f(x+y))+f(x-y)-f(x)-f(y), \quad x,y \in \mathbb{R}, \tag{18}$$
$$f(f(x)-f(y)) \leq f(f(x+y))+f(f(x-y))-f(f(x))-f(y), \quad x,y \in \mathbb{R}. \tag{19}$$

Observe that in the above problems the unknown function f appears also as an argument of f. Therefore, inequalities (17), (18) and (19) are so-called *composite functional inequalities.* It is worth to underline that the papers [F2] and [F6] are innovative in investigations of inequalities of this type. Classical methods are of no use in the case of composite functional inequalities.

For inequalities (17), (18) and (19) we have proved that if f is continuously differentiable with $f(0)=0$ and some estimate for the value $f'(0)$ holds true, then every solution is of one of the following forms: $f(x)=x$ or $f(x)=f'(0)x$ for $x \in \mathbb{R}$. Inequality (17) is a bit more difficult compared to (18) and (19) and in order to solve it we needed slightly stronger assumptions.

Hlawka's inequality. Application of Dini derivatives and Denjoy-Young-Saks theorem.

The inequality presented next, which can be also treated as a strengthening of the triangle condition, is Hlawka's inequality:

$$\|x+y\|+\|y+z\|+\|x+z\| \leq \|x+y+z\|+\|x\|+\|y\|+\|z\|. \tag{20}$$

Edmund Hlawka observed it for complex numbers, which was noted in the paper [10] of Hans Hornich in 1942. One can check that inequality (20) is not valid in every normed linear space. Any normed-linear space for which (20) holds true for all points x, y, z is called a *Hlawka space.* The problem of characterization of Hlawka spaces remains open, some partial results are recalled in the introduction of the article [F5].

In the paper [F5] we focus on the functional inequality stemming from (20):

$$f(x+y)+f(y+z)+f(x+z) \leq f(x+y+z)+f(x)+f(y)+f(z). \tag{21}$$

This inequality is investigated for real mappings defined on an Abelian group X and then for real-to-real functions. Observe that the particular solutions of (21) are functions: $f = \|\cdot\|$ defined on a Hlawka space and $f = \|\cdot\|^2$ defined on a Hilbert space. More generally, for any additive functional $a\colon X \to \mathbb{R}$ defined on a group X and for any additive operator $L\colon X \to Y$ with values in a Hlawka or a Hilbert space, respectively, the mappings:

$$X \ni x \mapsto f(x) = \|Lx\| + a(x) \in \mathbb{R} \tag{22}$$

and

$$X \ni x \mapsto f(x) = \|Lx\|^2 + a(x) \in \mathbb{R} \tag{23}$$

satisfy (21). We prove that if a function f satisfies (21) and vanishes at zero then its odd part is additive ([F5, Lemma 1]). Next, assuming one of the following homogeneity conditions:

$$\begin{aligned} f(2x) &= 2f(x), \\ f(2x) &= 3f(x) + f(-x), \\ f(2x) &= 4f(x), \end{aligned}$$

for all $x \in X$ we have proved that every solution of (21) is of the form (22), (23) or (23) with $a = 0$, respectively ([F5, Section 2]).

The problem of solving inequality (21) is much more difficult if we do not assume a homogeneity condition. However, we have obtained some satisfying results also in this situation. We restricted ourselves to real-to-real functions and we used the method of "differentiating inequalities side-by-side". Having in mind that we want to keep a high generality of our considerations, in particular we should avoid assuming high regularity of the function f, we decided to employ the Dini derivatives of f. In subsequent lemmas presented in the third paragraph of the paper [F4] we derive properties of upper and lower derivatives $D^{\pm}f$ and $D_{\pm}f$. Next, we show that functions $D_+f(x) + D^+f(-x)$ and $D_-f(x) + D^-f(-x)$ of a real variable x are bounded from both sides by expressions which depend only on values of the Dini derivatives of f at zero. Next, we prove subadditivity of the mappings $-D_+f + D^+f(0)\colon \mathbb{R} \to \overline{\mathbb{R}}$ and $D^-f - D_-f(0)\colon \mathbb{R} \to \overline{\mathbb{R}}$ with possibly infinite values (we use the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$). Then, we apply the Rosenbaum lemma (see R.A. Rosenbaum [17] or E. Hille, R.S. Philips [9, Theorem 7.3.3]) for subadditive functions taking values in $\overline{\mathbb{R}}$. This useful lemma says that if a subadditive function $\varphi\colon \mathbb{R} \to \overline{\mathbb{R}}$ is Lebesgue measurable and $\varphi(t_0) < +\infty$ at some negative point t_0, then either $\varphi = +\infty$ almost everywhere on $(0, +\infty)$, or φ is finite (everywhere) on $\mathbb{R}$. Informations obtained thanks to the Rosenbaum lemma are compared with the assertion of the Denjoy-Young-Saks theorem, which describes the possible behaviour of Dini derivatives for any real-to-real function. Thus we have proved that under quite weak assumptions the function f is differentiable almost everywhere and at least one of its Dini derivatives is finite at every point. The next step of the proof is to use a result of Zbigniew Gajda [5, Corollary 3.1] about the stability of the Cauchy functional equation almost everywhere. If we denote $B = \frac{1}{2}[D^+f(0) + D_-f(0)]$, $C = \frac{1}{2}[D^+f(0) - D_-f(0)]$ and define the function g by $g(t) = f'(t) - B$, then g is a well-defined function for almost all $t \in \mathbb{R}$ and the stability estimate

$$|g(x+y) - g(x) - g(y)| \leq C$$

holds true for almost all pairs $(x, y) \in \mathbb{R}^2$. Using Gajda's results we infer that the (existing almost everywhere) derivative of f has a uniform approximation almost everywhere:

$$|f'(x) - 2Ax - B| \leq C$$

for some constant A. A description of f is obtained by taking the integrals of both sides of the above estimate. Using the fact that since at least one of the Dini derivatives of f is finite everywhere, we can calculate its Henstock integral. But f is differentiable almost everywhere and the Henstock integral of f' coincides with its Lebesgue integral. In this way we obtained the following theorem which is the main result of the paper [F5].

Theorem 7 ([F5, Theorem 8]). *Assume that a function $f\colon \mathbb{R} \to \mathbb{R}$ is measurable with $f(0) = 0$, all its Dini derivatives are finite at zero and at least one of them is finite at some positive point and at some negative point and at least one of these derivatives is finite at a set of positive Lebesgue measure. If f satisfies (21) for all $x, y, z \in \mathbb{R}$, then there exist a constant $A \in \mathbb{R}$ and a mapping $r\colon \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = Ax^2 + Bx + r(x)$$

and

$$|r(x)| \leq C|x|$$

for all $x \in \mathbb{R}$, where $B = \frac{1}{2}[D^+f(0) + D_-f(0)]$ and $C = \frac{1}{2}[D^+f(0) - D_-f(0)]$.

If in the above theorem we assume additionally that $D^+f(0) \leq D_-f(0)$, then we get that f satisfies inequality (21) if and only if there exist constants $A, B \in \mathbb{R}$ such that $f(x) = Ax^2 + Bx$ for all $x \in \mathbb{R}$ ([F5, Corollary 5]).

It is worth to mention that the results of the paper [F5] have already achieved a considerable interest among the international specialist working in the field of functional equations and inequalities. During the conference *49th International Symposium on Functional Equations* in Graz-Mariatrost (Austria), where the author presented results concerning inequality (21), Zsolt Páles [15] posed several problems motivated by the talk of the habilitation candidate. Moreover, the results of the paper [F5] were presented with detailed proofs by the habilitation candidate in the Seminar of the Chair of Analysis at the University of Debrecen (Hungary) during his scientific visit in June 2012.

Averaging operators and related functional inequalities.

One of the earliest problems investigated connected with the *composite functional equations* is the averaging operators equation:

$$T(f \cdot Tg) = Tf \cdot Tg. \tag{24}$$

First it appeared in the thirties of the 20th century in works of Joseph Kampé de Fériet [11]. Later it was studied by Garrett Birkhoff [2] and by other mathematicians, also with multiplication replaced by an arbitrary group or semigroup operation. A linear operator $T\colon \mathcal{A} \to \mathcal{A}$ acting on an algebra $\mathcal{A}$ is called *averaging operator*, if it satisfies equation (24) for all functions f, g. The topic of the averaging operators was investigated intensively in the sixties and seventies of the last century. Presently, a few papers on related topics are also being published.

The aim of the paper [F6] is to study a modification of the above-mentioned problem with the equality sign in (24) replaced by an inequality sign. It is obvious that our considerations should be restricted to ordered structures, for example to the algebra $C(X)$ of all real functions on some compact set or, more generally, to ordered rings. In the context of earlier studies of (24) it is justified to focus on the following two inequalities:

$$T(f + T(g)) \geq T(f) + T(g), \tag{25}$$
$$T(f \cdot T(g)) \geq T(f) \cdot T(g). \tag{26}$$

Let us point out that we do not assume linearity of the operator T in the above problems. As a consequence we obtain a limited description of the solutions. An additional difficulty, which was pointed out when investigating problems (17), (18) and (19), is the character of inequalities (25) and (26) - we deal with the functional inequalities of composite type. Thus we do not have any selection of methods which allow us to find their solution in an effective way.

The paper [F6] begins with a sequence of examples of mappings satisfying inequality (25) or (26), which show that solutions can be far from the form one could expect if the range of an unknown function T is not at least an additive group or it is not a set of non-empty interior. Next, it is shown that if $\mathcal{R}$ is a partially ordered ring, then every solution $T\colon \mathcal{R} \to \mathcal{R}$ of (25) is of the form

$$T(f) = f + T(0) \tag{27}$$

for all elements f from the set $T(\mathcal{R}) \cap -T(\mathcal{R})$ ([F6, Lemma 2.1]). In particular, we know that all surjective solutions are of the form (27) for all $f \in \mathcal{R}$ ([F6, Corollary 2.1]). Further, it is proven that if the ring $\mathcal{R}$ is furnished with an additional topological structure, then solutions are also of this form under substantially weaker assumptions than surjectivity ([F6, Corollary 2.2]).

Since the ring $\mathcal{R}$ can contain zero divisors, inequality (26) is much more difficult to deal with. It is shown that if $\mathcal{R}$ is a unitary ring with $1 > 0$, then for all f from some subset of $\mathcal{R}$ the equality $T(f) = T(1) \cdot f$ holds ([F6, Lemma 2.3 and Theorem 2.1]).

Lax-Milgram lemma. An application of multifunctions and a selection theorem.

The last paper of the monographic set of publications deals with a modification of the Lax-Milgram lemma. The result proven has a weaker assertion, but it is obtained without the coercivity assumption. Moreover the bilinearity condition is replaced by a system of inequalities. The classical result of Peter Lax and Arthur Milgram published in [13] is usually formed in the following way:

Theorem 8 (P. Lax, A. Milgram [13]). *Assume that H is a real Hilbert space with the inner product $(\cdot|\cdot)$, $B\colon H \times H \to \mathbb{R}$ is a bilinear functional which is bounded and satisfies the* coercivity condition*:*

$$c\|u\|^2 \leq B(u,u) \tag{28}$$

for some constant $c > 0$ and for all $u \in H$. Then every continuous linear functional on H is of the form $B(\cdot, w)$ for some uniquely determined vector $w \in H$.

Equivalently, using the Riesz Representation theorem one can reformulate the assertion of the Lax-Milgram lemma in the following way:

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Functional B can be represented as

$$B(u,v) = (Tu, v), \quad u, v \in H,$$

where $T\colon H \to H$ is a continuous and bijective linear operator.

The main result of [F7] is the following theorem.

Theorem 9 ([F7, Theorem 5]). *Assume that H is a real Hilbert space and $B\colon H \times H \to \mathbb{R}$ is a mapping which satisfies the following conditions:*

(a) *for every $u \in H$ the map $B(u, \cdot)$ is sublinear;*

(b) *for every $v \in H$ the map $B(\cdot, v)$ is superadditive;*

(c1) *if $B(u,v) \geq 0$ for some $u \in H$ and for each $v \in H$, then $u = 0$;*

(c2) *if $B(u,v) = 0$ for each $u \in H$ and for some $v \in H$, then $v = 0$;*

(d) *B is bounded from above on some non-empty open subset of $H \times H$.*

Then, there exists a bounded linear operator $T\colon H \to H$ such that:

(i) *T is injective;*

(ii) *the image of T is dense in H;*

(iii) *$(Tu|v) \leq B(u,v)$ for all $u, v \in H$.*

In the proof of Theorem 9 we use a result of Ger from the paper [7], which says that every continuous sublinear function defined on H has the following representation:

$$f(x) = \sup\{(a|x) : a \in K\}, \quad x \in H,$$

where K is a non-empty weakly compact and convex set. Moreover, the set K is determined uniquely as

$$K = \{x \in H : (x|v) \leq f(v) \text{ for } \quad \text{all } v \in H\}.$$

Using these two facts we construct some multifunction $m\colon H \to \mathrm{cc}(H)$. Next, we show the relation

$$m(u) + m(v) \subseteq m(u+v)$$

Kw

for every $u, v \in H$. Further, using the selection theorem for superadditive multifunctions due to Gajda [5, Theorem 4.4] we obtain an additive selection of the multifunction m. Concluding the proof we show that this selection (which is equal to the postulated operator T) satisfies conditions (i), (ii), (iii) of Theorem 9.

It is worth to emphasize that the very desirable property of multifunctions that its values are non-empty, compact and convex sets is in our case a consequence of a suitable functional inequality. Therefore, we believe the same effect is true also for a broader class of problems. If this is the case, then selection theorems could be applied in numerous problems in the theory of functional inequalities.

An undoubted drawback of the results of this section is the lack of presently known applications. The basic problem which precludes us from applying Theorem 9 in an analogical way as the Lax-Milgram lemma is connected with too weak properties of the operator T provided in Theorem 9. Namely, we have not proved the bijectivity of T. The problem seems to lie in the fact that in our case the coercivity assumption (28) does not provide the same effects if the bilinearity assumption is replaced by a system of inequalities. In standard proofs of the Lax-Milgram theorem the condition (28) forces the bijectivity of T. In our problem an analogous implication is not true. At this moment the author does not know any condition which together with the other assumptions of Theorem (9) would imply the assertion similar to the Lax-Milgram lemma.

The results of this paragraph were presented in February 2012 by the author on the seminar *Ulmer Seminare "Funktionalanalysis und Differentialgleichungen"* presided over by Wolfgang Arendt, where the talk received a considerable interest. During the discussion after the talk Markus Kunze pointed out some connections of the presented results with the Bellman equation and with similar problems investigated e.g. in optimization theory (cf. also a survey [1]). Moreover, in June 2012 the habilitation candidate presented the same result on the conference *The Fiftieth International Symposium on Functional Equations* in Hajdúszoboszló (Hungary). The talk was awarded the medal *For Outstanding Contribution* granted by the Scientific Committee of the conference.

The habilitation candidate plans to conduct further research on this topic, also using other selection theorems, as for example Michael's theorem which gives conditions under which a lower semicontinuous multifunction has a continuous selection.

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5. A summary of the remaining scientific achievements.

   (a) List of other (not included in the scientific achievement spoken of in point 5) published scientific papers.

      (A) Scientific papers in journals indexed in the database Journal Citation Reports (Edition 2011):

[F8] Włodzimierz Fechner, *On functions with the Cauchy difference bounded by a functional. Part III*, Abh. Math. Sem. Univ. Hamburg **76** (2006), 57–62.

[F9] Włodzimierz Fechner, *Stability of a functional inequality associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.

[F10] Włodzimierz Fechner, *On the Hyers-Ulam stability of functional equations connected with additive and quadratic mappings*, J. Math. Anal. Appl. **322** (2006), 774–786.

[F11] Włodzimierz Fechner, *Characterization of quadratic mappings through a functional inequality*, J. Math. Anal. Appl. **324** (2006), 452–459.

[F12] Włodzimierz Fechner, *On a functional inequality connected with quadratic functionals*, J. Math. Anal. Appl. **332** (2007), 381–389.

[F13] Włodzimierz Fechner, *On an abstract version of a functional inequality*, Math. Inequal. Appl. **11/2** (2008), 381–392.

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[F15] Włodzimierz Fechner, *On some functional inequalities related to the logarithmic mean*, Acta Math. Hungar. **128/1-2** (2010), 36–45.

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[F17] Włodzimierz Fechner, *Functional equations with exotic addition*, Nonlinear Anal. **74/17** (2011), 5999–6003.

[F18] Włodzimierz Fechner, *A characterization of quadratic-multiplicative mappings*, Monatsh. Math. **164/4** (2011), 383–392.

[F19] Włodzimierz Fechner, Eszter Gselmann, *General and alien solutions of a functional equation and of a functional inequality*, Publ. Math. Debrecen **80/1–2** (2012), 143–154 (the exact contribution is described in the co-author acknowledgement).

[F20] Włodzimierz Fechner, Justyna Sikorska, *On a separation for the Cauchy equation on spheres*, Nonlinear Anal. **75** (2012), 6306—6311 (the exact contribution is described in the co-author acknowledgement).

[F21] Włodzimierz Fechner, *On a composite functional equation satisfied almost everywhere*, Indagationes Math. **24/1** (2013), 103—110.

[F22] Włodzimierz Fechner, *Quadratic operators on AM-spaces*, Glasnik Mat. (accepted).

(B) Scientific papers in journals other than indexed in the database Journal Citation Reports (Edition 2011).

[F23] Włodzimierz Fechner, *On functions with the Cauchy difference bounded by a functional*, Bull. Polish Acad. Sci. Math. **52/3** (2004), 265–271.

[F24] Włodzimierz Fechner, *On functions with the Cauchy difference bounded by a functional. Part II*, Int. J. Math. Sci. **2005/12** (2005), 1889–1898.

[F25] Włodzimierz Fechner, *Separation theorems for conditional functional equations*, Ann. Math. Sil. **21** (2007), 31–40.

[F26] Włodzimierz Fechner, *Some inequalities connected with the exponential function*, Arch. Math. (Brno) **44** (2008), 217–222.

[F27] Włodzimierz Fechner, Justyna Sikorska, *Sandwich theorems for orthogonally additive functions*, C. Bandle et al. (eds.), Inequalities and Applications 2007, International Series of Numerical Mathematics, **157** (2009), 269–281 (the exact contribution is described in the co-author acknowledgement).

[F28] Włodzimierz Fechner, Justyna Sikorska, *On the stability of orthogonal additivity*, Bull. Polish Acad. Sci. Math. **58** (2010), 23–30 (the exact contribution is described in the co-author acknowledgement).

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[F30] Włodzimierz Fechner, *Functional equations motivated by the Lagrange's identity*, Demonstratio Math. **44/1** (2011), 91–98.

[F31] Włodzimierz Fechner, *On some functional-differential inequalities related to the exponential mapping*, Tokyo J. Math. **34/2** (2011), 345–352.

[F32] Włodzimierz Fechner, *Functional inequalities and equivalences of some estimates*, C. Bandle et al. (eds.), Inequalities and Applications 2010, dedicated to the Memory of Wolfgang Walter. International Series of Numerical Mathematics **161** (2012), 231–240.

(b) A description of the scientific output of the habilitation candidate obtained after the doctoral degree and which is not included in the monographic set of publications:

Other scientific output of the habilitation candidate concerns broadly understood theory of functional equations and inequalities jointly with its connections with other branches of mathematics. One can separate a few groups. Papers [F9], [F10], [F13], [F16], [F28], [F29] deal with the Hyers-Ulam stability of functional equations and inequalities. These problems are currently intensively investigated by a number of mathematicians and are connected with other areas of mathematics. Let us emphasize that the paper [F9], which is devoted to the stability of the following functional inequality:

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\|,$$

is the most frequently cited paper of the habilitation candidate and presently has 50 citations (without self-citations) according to Google Scholar database.

The papers [F15], [F26], [F32] are devoted to the topic of comparisons of means and related functional inequalities. Let the letters A, G and L denote the arithmetic, geometric and logarithmic mean, respectively. The following estimates are well known:

$$G(s,t) \le L(s,t) \le A(s,t), \tag{29}$$

and

$$G^{\frac{2}{3}}(s,t) \cdot A^{\frac{1}{3}}(s,t) \le L(s,t) \le \frac{2}{3}G(s,t) + \frac{1}{3}A(s,t) \tag{30}$$

for all $s,t > 0$.

Let us fix arbitrary $x, y \in \mathbb{R}$ such that $x \neq y$ and substitute $s := e^x$ and $t := e^y$ in (29) and (30). Therefore, we see that the exponential function satisfies the following inequalities:

$$e^{\frac{x+y}{2}} \le \frac{e^y - e^x}{y-x} \le \frac{e^x + e^y}{2} \tag{31}$$

and

$$6e^{\frac{2}{3}\cdot\frac{x+y}{2}} \left[\frac{e^x + e^y}{2}\right]^{\frac{1}{3}} \le 6\frac{e^y - e^x}{y-x} \le 4e^{\frac{x+y}{2}} + e^x + e^y. \tag{32}$$

Both foregoing estimates are motivation for the studies of the habilitation candidate published in the above mentioned papers of the following four functional inequalities:

$$f\left(\frac{x+y}{2}\right) \le \frac{f(y) - f(x)}{y-x},$$

$$\frac{f(y) - f(x)}{y-x} \le \frac{f(x) + f(y)}{2},$$

$$f\left(\frac{x+y}{2}\right)^2 \cdot \frac{f(x) + f(y)}{2} \le \left[\frac{f(y) - f(x)}{y-x}\right]^3,$$

$$6\frac{f(y) - f(x)}{y-x} \le 4f\left(\frac{x+y}{2}\right) + f(x) + f(y),$$

for x, y belonging to an interval I. Under some assumptions solutions of above inequalities are of the form

$$f(x) = m(x) \cdot \exp(x), \quad x \in I,$$

where $m \colon I \to \mathbb{R}$ is a monotone mapping.

In the articles [F20], [F25], [F27] problems of separation by a function and sandwich theorems for conditional functional equations are studied. Assume that X is a non-empty set and we are given some binary relation $\mathcal{R}$ on X. Further, let $p \colon X \to \mathbb{R}$ and

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$q\colon X \to \mathbb{R}$ be two functions such that

$$\mathcal{R}(x,y) \Longrightarrow p(x+y) \leq p(x) + p(y),$$
$$\mathcal{R}(x,y) \Longrightarrow q(x+y) \geq q(x) + q(y).$$

By the term *sandwich theorem* we mean a statement which provide conditions sufficient for the existence of a function $f\colon X \to \mathbb{R}$ which solve the functional equation

$$\mathcal{R}(x,y) \Longrightarrow f(x+y) = f(x) + f(y)$$

and $p \leq f \leq q$ or $q \leq f \leq p$.

In the joint paper with Justyna Sikorska [F27] we obtained sandwich theorems for the orthogonal additivity, i.e. we dealt with the case when $\mathcal{R}$ is the orthogonality relation on an inner product space or on a more general structure. Therefore, we proved that orthogonally subadditive and orthogonally superadditive mapping can be separated by an orthogonally additive function.

In the papers [F20] and [F27] we considered the situation when X is a normed linear space and $\mathcal{R}(x,y)$ holds when $\|x\| = \|y\|$. Also, more general settings have been discussed. We provided conditions under which mappings which are subadditive or superadditive, respectively, on vectors of equal length can be separated by an additive function.

The paper [F14] is devoted to the following composite functional equation:

$$f(f(x) - f(y)) = f(x+y) + f(x-y) - f(x) - f(y),$$

for an unknown mapping $f\colon G \to G$ acting on an Abelian group $(G,+)$. Results of this paper also motivated in part our subsequent studies published in the articles [F2] and [F16].

In the paper [F17] the habilitation candidate dealt with binary operations of the form:

$$x \oplus y := xf(y) + yf(x)$$

on an arbitrary interval containing zero. Moreover, Jensen functional equation and equation of derivations with the ordinary addition replaced by the operation $\oplus$ have been solved.

The paper [F18] is devoted to the alienation problem for quadratic and multiplicative mappings. Conditions are provided under which solutions of the following equation:

$$af(xy) + bf(x)f(y) + cf(x+y) + df(x-y) + k(f(x) + f(y)) = 0$$

are quadratic-multiplicative mappings. This result is also connected with the paper [F4] of the monographic set of publications.

The purpose of the paper [F19] written jointly with Eszter Gselmann was to solve the equation

$$g(x+y)-g(x)-g(y)=xf(y)+yf(x).$$

After determining the general solutions, we considered the alienation problem, i.e. we examined the cases when the above equation implies that

$$g(x+y)=g(x)+g(y)$$

and

$$xf(y)+yf(x)=0.$$

Finally, we discussed a related functional inequality:

$$g(x+y)-g(x)-g(y)\geq xf(y)+yf(x).$$

The paper [F30] is devoted to the following two functional equations:

$$f\left(\sum_{i=1}^{n}a_ib_i\right)=\left(\sum_{i=1}^{n}f(a_i)\right)\left(\sum_{i=1}^{n}f(b_i)\right)-\sum_{1\leq i<j\leq n}f(a_ib_j-a_jb_i)$$

and

$$f\left(\sum_{i=1}^{n}a_ib_i\right)=\left(\sum_{i=1}^{n}f(a_i)\right)\left(\sum_{i=1}^{n}f(b_i)\right)-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}f(a_ib_j-a_jb_i).$$

Both equations are motivated by the Lagrange identity, which states that for every positive integer n and each a_i, b_i from a commutative ring R, where $i=1,\dots,n$ we have

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2=\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)-\sum_{1\leq i<j\leq n}(a_ib_j-a_jb_i)^2$$

or equivalently

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2=\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j-a_jb_i)^2.$$

In the paper [F31] the habilitation candidate considered real-valued twice differentiable functions defined on an open interval. The main result of this paper states that if a function f is a solution of some functional-differential inequalities then a map $x \mapsto f(x)\exp(-cx)$ is convex, where c is an arbitrary point of $\mathbb{R} \setminus (c_1, c_2)$ for some real c_1, c_2.

The last group is the paper [F22] jointly with some manuscripts of the habilitation candidate which are in preparation or under review and concerns some problems of the operator theory. In particular the habilitation candidate obtained some analogues of representation theorems of linear operators on AM-spaces for quadratic operators. What is more, generalizations of factorization theorems of Arendt have been proven (extensions of the Luxemburg-Schep theorem and its dual theorem, which are operator versions of the Radon-Nikodym theorem). It is planned that this studies will be continued jointly with Tomasz Kochanek within a framework of the project *Linear and nonlinear factorizations of operators, together with their stability properties in C^*-algebras and in lattices*, which is accepted for being supported by by the Polish Ministry of Science and Higher Education in a framework of the program Iuventus Plus 2012.

Włodzimierz Fechner