ON A DEFINITION OF MORSE-SMALE
EVOLUTION PROCESSES

Radosław Czaja

Institute of Mathematics
University of Silesia
Bankowa 14, 40-007 Katowice, Poland

Waldyr M. Oliva[1,2] and Carlos Rocha[*,2]

[1] Instituto de Matemática e Estatística, Universidade de São Paulo
Rua do Matão, 1010, 05508-090 São Paulo, Brazil
[2] Center for Mathematical Analysis, Geometry, and Dynamical Systems
Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa
Av. Rovisco Pais, 1049-001 Lisboa, Portugal

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Abstract. In this paper we consider a definition of Morse-Smale evolution process that extends the notion of Morse-Smale dynamical system to the nonautonomous framework. In particular we consider nonautonomous perturbations of autonomous systems. In this case our definition of Morse-Smale evolution process holds for perturbations of Morse-Smale autonomous systems with or without periodic orbits. We establish that small nonautonomous perturbations of autonomous Morse-Smale evolution processes derived from certain nonautonomous differential equations are Morse-Smale evolution processes. We apply our results to examples of scalar parabolic semilinear differential equations generating evolution processes and possessing periodic orbits.

1. **Introduction.** In recent years there is an increasing interest in the study of nonautonomous evolution equations expressed either by ordinary or partial differential equations. In the mathematical literature the main object of these studies is the long time behavior of the solutions of these systems, see [11, 2, 30, 8]. The appropriate notions of attractor in the nonautonomous framework have been considered, and the study of their continuity and structural stability properties is still going on. Here the study of nonautonomous perturbations of autonomous systems is of particular relevance. Bortolan, Carvalho and Langa addressed these questions in [4]. They introduce a notion of Morse-Smale evolution process which is suitable for dynamically gradient evolution processes. This approach, reminiscent of Conley's treatment of isolated invariant sets, [15], is quite appropriate for gradient-like flows. However, if the unperturbed autonomous system contains periodic orbits

this approach is not adequate unless we consider the collapse of the corresponding invariant sets. Since the classical notion of Morse-Smale dynamical system (in the autonomous framework) also involves the case of (finitely many hyperbolic) periodic orbits, it is appropriate to consider nonautonomous perturbations of autonomous systems possessing periodic orbits. In this paper we consider a definition of Morse-Smale evolution process that extends the notion of Morse-Smale dynamical system to the nonautonomous framework.

A Morse-Smale dynamical system $\phi_0 : \mathbb{R} \times \mathcal{B} \mapsto \mathcal{B}$ has a finite number of hyperbolic equilibria and also a finite number of hyperbolic periodic orbits. Let $S_0(\cdot, \cdot)$ denote the corresponding autonomous evolution process in $\mathcal{B}$ and consider the nonautonomous evolution processes $S_\varepsilon(\cdot, \cdot)$ corresponding to its (nonautonomous) perturbations. Under a small ε-perturbation the solutions corresponding to the hyperbolic equilibria of ϕ_0 become hyperbolic solutions of the nonautonomous evolution process $S_\varepsilon(\cdot, \cdot)$, [5]. But to understand the nonautonomous perturbations of a hyperbolic periodic orbit we need to consider the corresponding (normally hyperbolic) isolated invariant manifolds. This is illustrated in Section 2 by a simple example of an ordinary differential equation on $\mathcal{B} = \mathbb{R}^2$. We consider nonautonomous perturbations of an autonomous ODE with one hyperbolic periodic orbit and describe the perturbations of the isolated invariant cylinder in $\mathbb{R} \times \mathcal{B}$ corresponding to the periodic solutions of the unperturbed system. In particular, we illustrate the existence of two special types of perturbations (normal and tangential) for these invariant cylinders.

In Section 3 we recall the notions of semiflows, global orbits, invariant sets and global attractors. We also recall the extension of these notions to the nonautonomous setting and we use the induced semiflows on $\mathbb{R} \times \mathcal{B}$ to compare the evolution processes.

The central notion of pullback attractor and its continuity properties is considered in Section 5.

In Section 6 we recall the notions of exponential dichotomy and trichotomy associated to hyperbolic and partially hyperbolic solutions of evolution process. We consider evolution process admitting a pullback attractor and satisfying well-known reversibility properties. We also recall the notions of stable and unstable manifolds and consider the normally hyperbolic isolated invariant manifolds associated to (perturbations of) hyperbolic periodic orbits.

In Section 7 we recall and collect the classic results on the behavior of hyperbolic and normally hyperbolic isolated invariant manifolds under perturbation. The study of conditions which ensure that such invariant manifolds are preserved under perturbations are abundant in the literature. See for example Sacker [36], Fenichel [18] and Hirsch, Pugh and Shub [27]. Quoting Pliss and Sell, [35]: "The fact that a normally hyperbolic invariant manifold persists under a small C^1-perturbation is, of course, well known". Here we collect some results on the behavior of certain isolated invariant manifolds for differential equations under nonautonomous perturbations. We essentially recall the classic results on integral manifolds for perturbed differential systems (the Krylov-Bogoliubov method) following Bogoliubov and Mitropolski [3], Hale [20, 21, 22], Coppel and Palmer [16], and Henry [25].

The definition of Morse-Smale systems on $\mathcal{B}$ involves the concept of nonwandering behavior for hyperbolic critical orbits. For the case of processes, we instead introduce in Section 8 the notion of recurrent behavior on $\mathbb{R} \times \mathcal{B}$ neighborhoods of graphs of global solutions $z(\cdot)$ (simply referred to as recurrent trajectories). We also introduce the notion of recurrent behavior outside a manifold $\mathcal{M}_z$. This notion is

motivated by the behavior of normally hyperbolic invariant manifolds corresponding to periodic orbits under normal/tangential perturbations as considered in Section 2.

Using the notion of recurrent behavior we define Morse-Smale evolution process in Section 9. Then we show in Theorem 9.2 that autonomous evolution processes corresponding to Morse-Smale semiflows on $\mathcal{B}$ are Morse-Smale evolution processes. Subsequently we consider evolution processes obtained by nonautonomous perturbations of Morse-Smale autonomous evolution processes. We specifically study processes derived from certain nonautonomous differential equations already analyzed in Section 7 for the persistence of isolated invariant manifolds. For these problems we establish in Theorem 9.3 the following openness property: small nonautonomous perturbations of autonomous Morse-Smale evolution processes are Morse-Smale evolution processes.

In the last Section we apply these results to examples of nonautonomous perturbations of autonomous Morse-Smale systems. In particular we consider scalar parabolic semilinear differential equations generating evolution processes on adequate fractional power spaces and possessing periodic orbits. We also consider the case of asymptotically autonomous evolution processes with a Morse-Smale limiting behavior.

The relevance of Morse-Smale systems for the discussion of structural stability of autonomous systems is well established in the literature. For nonautonomous systems, structural stability has also been considered; see for example [31] in the finite dimensional case, and [2] for local aspects in infinite dimensions.

The notion of Morse-Smale evolution process for a gradient like system introduced in [4] proved useful for the discussion of qualitative aspects of the dynamics related to structural stability of nonautonomous systems (see [9]). We believe that the notion of Morse-Smale evolution process introduced here is appropriate for the discussion of structural stability, but this topic should be addressed elsewhere.

2. **Nonautonomous perturbations of autonomous ODEs.** In this Section we describe examples of nonautonomous perturbations of an autonomous ODE with one hyperbolic periodic orbit. We consider the perturbations of the isolated invariant cylinder corresponding to the periodic solutions of the unperturbed system. We show that, while certain perturbations deform the cylinder into a tube fibered by nonhyperbolic trajectories (a normally hyperbolic invariant manifold, see Sections 6-7), other perturbations preserve the cylinder but perturb the flow on it producing hyperbolic behavior.

Let us start with the following example of a nonautonomous system of ordinary differential equations on $\mathcal{B} = \mathbb{R}^2$,

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) + \delta f_1(t, x, y) \\ \dot{y} = x + y(1 - x^2 - y^2) + \delta f_2(t, x, y) \end{cases} \tag{1}$$

where δ is a real parameter and f_1, f_2 are sufficiently smooth bounded functions. In polar coordinates, $x = \rho \cos \theta, y = \rho \sin \theta$, system (1) is given as

$$\begin{cases} \dot{\rho} = \rho(1 - \rho^2) + \delta(f_1 \cos \theta + f_2 \sin \theta) \\ \rho \dot{\theta} = \rho + \delta(-f_1 \sin \theta + f_2 \cos \theta) \ , \end{cases} \tag{2}$$

with (f_1, f_2) also in polar coordinates.

For $\delta = 0$ we obtain the unperturbed autonomous system

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \ . \end{cases} \tag{3}$$

This system possesses only one fixed point $(0,0)$ and one stable limit cycle

$$\Pi = \{(x,y) \in \mathcal{B} : x^2 + y^2 = 1\} \ .$$

Moreover, from the linearization of (3) around periodic solutions, it is known that $\mu = 1$ is a simple characteristic multiplier and Π is a hyperbolic periodic orbit. In $\mathbb{R} \times \mathcal{B}$ the set of trajectories of periodic solutions of (3) forms a cylinder with Π as its natural projection on $\mathcal{B}$.

For small $\delta \neq 0$ we first consider a nonautonomous perturbation of the form

$$(f_1(t,x,y), f_2(t,x,y)) = \varphi(x^2 + y^2)g(t)\frac{(x,y)}{\sqrt{x^2 + y^2}} \ , \tag{4}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth bounded function and $\varphi : \mathbb{R}^+ \to [0,1]$ is a smooth cutoff satisfying

$$\varphi(r) = \begin{cases} r^2, \ 0 \leq r \leq \frac{1}{4} \\ 1, \ \frac{1}{2} \leq r \leq 2 \\ 0, \ r \geq 4 \ , \end{cases}$$

and monotone for $r \in (1/4, 1/2)$ and $r \in (2,4)$. In polar coordinates this is

$$(f_1, f_2) = \varphi(\rho^2)g(t)(\cos\theta, \sin\theta) \ , \tag{5}$$

which leads to the perturbed system

$$\begin{cases} \dot{\rho} = \rho(1 - \rho^2) + \delta\varphi(\rho^2)g(t) \\ \dot{\theta} = 1 \ . \end{cases} \tag{6}$$

This is a simpler case of the example in [4, Section 3.4] where it is pointed out that this perturbation deforms the cylinder into a tube containing an infinite number of trajectories of nonhyperbolic global solutions. In view of this deformation we say that a perturbation of the form (4) is a *normal perturbation* with respect to the invariant cylinder.

Our next example uses a perturbation of the form

$$(f_1(t,x,y), f_2(t,x,y)) = \varphi(x^2 + y^2)(x\sin t - y\cos t)\frac{(y,-x)}{\sqrt{x^2 + y^2}} \ , \tag{7}$$

which in polar coordinates is

$$(f_1, f_2) = \rho\varphi(\rho^2)\sin(t - \theta)(\sin\theta, -\cos\theta) \ , \tag{8}$$

leading to the perturbed system

$$\begin{cases} \dot{\rho} = \rho(1 - \rho^2) \\ \dot{\theta} = 1 + \delta\varphi(\rho^2)\sin(\theta - t) \ . \end{cases} \tag{9}$$

As a consequence, the perturbation preserves the invariant cylinder corresponding to the periodic orbit,

$$\mathbb{R} \times \Pi = \{(t,x,y) \in \mathbb{R} \times \mathcal{B} : x^2 + y^2 = 1\} \ ,$$

but changes completely the dynamics on the cylinder. In fact, for $\delta \neq 0$ the perturbed flow on the cylinder contains exactly two trajectories of hyperbolic solutions of (1). To see this, we change our polar coordinates to a co-rotating frame with

respect to the periodic orbit, $(\rho, \theta) \mapsto (\rho, \psi)$ with $\psi = \theta - t$. This substitution changes only the second equation of (9) and we obtain the autonomous system in the new polar coordinates

$$\begin{cases} \dot{\rho} = \rho(1 - \rho^2) \\ \dot{\psi} = \delta\varphi(\rho^2)\sin\psi \ . \end{cases} \tag{10}$$

An easy integration shows that the solutions have the form

$$\rho(t) = \begin{cases} \left[1 - (1 - 1/\rho_0^2)e^{-2t}\right]^{-1/2} & \text{if } \rho_0 \neq 0 \ , \\ 0 \ \text{if } \rho_0 = 0 \ , \end{cases}$$

$$\psi(t) = \begin{cases} 2\arctan\left(\tan(\psi_0/2)\exp(\delta\int_0^t \varphi(\rho^2(s))ds)\right) \tag{11} \\ \qquad \text{if } \psi_0 \in (-\pi, \pi) \ (\text{mod } 2\pi) \ , \\ \pi \ \text{if } \psi_0 = \pi \ (\text{mod } 2\pi) \ . \end{cases}$$

We observe that if $\rho_0 = 1$, then $\rho(t) = 1$ for all $t \in \mathbb{R}$ and (with the usual (mod 2π) polar identifications) we can distinguish three equilibria of (10), namely $(0,0), (1,0)$, and $(1,\pi)$. In the coordinates (ρ, θ) these equilibria correspond to the solutions $\xi_0(t) = (0,t), \xi_1(t) = (1,t)$, and $\xi_2(t) = (1, \pi + t)$ of (9). In the original coordinates (x, y) they correspond to the solutions $\zeta_0(t) = (0,0), \zeta_1(t) = (\cos t, \sin t)$, and $\zeta_2(t) = (-\cos t, -\sin t)$ of (1).

For $\delta \neq 0$ the three equilibria of the autonomous equation (10) are hyperbolic. The origin is a source (Morse index 2) and the remaining equilibria, one is a saddle (Morse index 1) and the other is a sink (Morse index 0). Moreover, the two equilibria $(1,0)$ and $(1,\pi)$ exchange their stability with the sign of δ. In addition, the phase diagram of (10) shows the equilibria $(1,0)$ and $(1,\pi)$ connected by two orbits completing the circle of radius 1, see Figure 1. These orbits run from $(1,0)$ to $(1,\pi)$ if $\delta > 0$ and from $(1,\pi)$ to $(1,0)$ if $\delta < 0$.
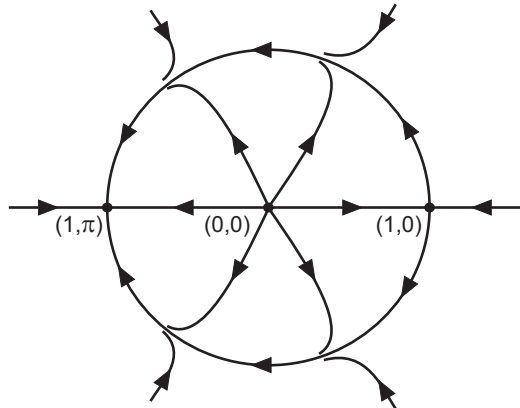

FIGURE 1. Phase portrait of the autonomous equation (10) for $\delta > 0$.

These facts indicate that, if $\delta \neq 0$, the solutions ζ_1 and ζ_2 of (1) are hyperbolic, see [2, Section 4.3], [8, Definition 8.1]. Indeed, additional computations show that the linearizations around these solutions admit exponential dichotomies with exponent $\beta = \min(\delta, 2)$.

Each bounded global orbit of the planar autonomous system (10) is either an equilibrium or a connecting orbit between two different equilibria. Since the equilibria have different Morse indices the solutions of (1) corresponding to the connecting orbits cannot be hyperbolic. Therefore, the nonautonomous system (1) admits only three hyperbolic solutions, ζ_0, ζ_1 and ζ_2.

In view of these observations we say that a perturbation of the form (7) is a *tangential perturbation* with respect to the invariant cylinder.

These examples illustrate two types of perturbations of invariant cylinders corresponding to periodic orbits: the normal perturbations which deform the cylinders into tubes fibered by an infinite number of trajectories of nonhyperbolic solutions, and the tangential perturbations which reduce the cylinders corresponding to periodic orbits to a finite set of trajectories of hyperbolic solutions and their heteroclinics. Therefore a definition of Morse-Smale nonautonomous system that takes into consideration the periodic orbits of autonomous systems has to comprise both types of perturbations.

3. **Semiflows and evolution processes.** We start by recalling the well-known notion of a semiflow.

Definition 3.1. Let E be a Banach space and let $C^k(E, E), k \geq 1$, be the set of all bounded C^k transformations of E, not necessarily injective, with bounded derivatives up to the order k. A C^k *semiflow on* E is a map

$$\mathbb{R}^+ \ni s \mapsto \mathcal{T}(s) \in C^k(E, E)$$

such that $\mathcal{T}(0) = \mathrm{Id} \in C^k(E, E)$ and

$$\mathcal{T}(s_1) \circ \mathcal{T}(s_2) = \mathcal{T}(s_1 + s_2), s_1, s_2 \geq 0.$$

One assumes also that the map

$$\mathbb{R}^+ \times E \ni (s, x) \mapsto \mathcal{T}(s)x \in E$$

is continuous.

For a semiflow one can define the notions of *global orbits*, *invariant sets* and *global attractors*, see e.g. [23], [38], [13]. The qualitative theory of dynamical systems involves the study of these global attractors and one of the essential tools available is the *Morse-Smale property* (see [34], [24] and [32]). Most of these notions introduced in the case of semiflows have been recently extended to the case of nonautonomous systems with the consideration of evolution processes and also skew product flows, see in particular [2], [30], [8] and the references therein. Here we will stay in the framework of evolution processes.

Let Δ be the diagonal of $\mathbb{R} \times \mathbb{R}$ and consider $\Delta_+ = \{(t, \tau) \in \mathbb{R} \times \mathbb{R} \colon t \geq \tau\}$ which is a manifold with boundary Δ and, moreover, is a groupoid acting on $\mathbb{R}$. Furthermore, let $\mathcal{F}$ be a given metric space of parameters, which in applications will be suitably chosen depending on the problem in view.

Definition 3.2. A C^k *evolution process* on a Banach space $\mathcal{B}$ parameterized by an element $f \in \mathcal{F}$ is a mapping S_f from Δ_+ into $C^k(\mathcal{B}, \mathcal{B})$ and such that
(i) $S_f(t, \sigma) \circ S_f(\sigma, \tau) = S_f(t, \tau)$ for all $t \geq \sigma \geq \tau$,
(ii) $S_f(\tau, \tau) = \mathrm{Id} \in C^k(\mathcal{B}, \mathcal{B})$ for all $\tau \in \mathbb{R}$,
(iii) the map

$$\Delta_+ \times \mathcal{B} \ni ((t, \tau), x) \mapsto S_f(t, \tau)x \in \mathcal{B}$$

is continuous.

Remark 1. Note that a nonautonomous differential equation on $\mathcal{B}$, satisfying conditions of existence, uniqueness and global continuation of solutions, defines an evolution process on $\mathcal{B}$.

Definition 3.3. The map $S\colon \mathcal{F} \times \Delta_+ \times \mathcal{B} \to \mathcal{B}$ given by

$$S(f,(t,\tau),x) := S_f(t,\tau)x \ ,$$

called the *global evolution process*, is assumed to be continuous and it is used to analyze the dependence on $f \in \mathcal{F}$ for a given family of evolution processes.

Definition 3.4. Let $f \in \mathcal{F}$ be such that a C^k evolution process S_f satisfies $S_f(t,\tau) = T_f(t-\tau)$, $t \geq \tau$, for a certain family $T_f(s)$, $s \geq 0$. It is easy to see that T_f is a semiflow of C^k transformations of $\mathcal{B}$. In this case the evolution process S_f is called an *autonomous evolution process*. Otherwise, S_f is said to be *nonautonomous*. Conversely, if $T_f(s)$, $s \geq 0$, is a semiflow of C^k transformations of $\mathcal{B}$, then by defining $S_f(t,\tau) := T_f(t-\tau)$ it can be seen that S_f is an autonomous evolution process on $\mathcal{B}$ (see [5], [10], [6]).

Definition 3.5. A continuous function $z\colon \mathbb{R} \to \mathcal{B}$ is called a *global solution* for $S_f(\cdot,\cdot)$ if one has $S_f(t,\tau)z(\tau) = z(t)$ for all $t \geq \tau$, $t,\tau \in \mathbb{R}$. A family of sets $\{A_f(t) \subset \mathcal{B},\ t \in \mathbb{R}\}$, is said to be *invariant under* $S_f(\cdot,\cdot)$ if $S_f(t,\tau)A_f(\tau) = A_f(t)$ for all $t \geq \tau$.

Definition 3.6. Under appropriate smoothness conditions of the global evolution process $S(f,(t,\tau),x)$ we define its derivative with respect to t on the diagonal Δ

$$D_1[S_f(t,\tau)x]_{t=\tau} = \lim_{s \to 0^+} \frac{S_f(\tau+s,\tau)x - x}{s}$$

with (τ,x) from a suitable domain $\mathcal{D}_\Delta(S) \subset \mathbb{R} \times \mathcal{B}$ depending on the particular problem in view. This domain $\mathcal{D}_\Delta(S)$, in the applications we envisage, will contain the set corresponding to the global solutions.

For instance, if we consider evolution processes generated by ODEs on $\mathcal{B} = \mathbb{R}^n$, then $\mathcal{D}_\Delta(S) = \mathbb{R} \times \mathbb{R}^n$. For evolution processes generated by delay differential equations, where $\mathcal{B} = C([-r,0],\mathbb{R}^n)$ we have $\mathcal{D}_\Delta(S) = \mathbb{R} \times \mathcal{B}$.

4. **Comparing evolution processes.** In order to compare evolution processes, in particular autonomous with nonautonomous processes and if, moreover, we want to analyze some stable structures that appear in nonautonomous evolution processes on $\mathcal{B}$, we will see in the sequel that any C^k evolution process on $\mathcal{B}$ induces a C^k semiflow on $\mathbb{R} \times \mathcal{B}$.

For instance, the equation (1) for $\delta = 0$ defines a Morse-Smale semiflow on $\mathcal{B} = \mathbb{R}^2$. For $\delta > 0$, the perturbed equation defines a nonautonomous process in the same space. Both processes can be considered and compared as semiflows on $\mathcal{B}_1 = \mathbb{R} \times \mathbb{R}^2$ generated by the equation

$$\begin{cases} t' = 1 \\ x' = -y + x(1-x^2-y^2) + \delta f_1(t,x,y) \\ y' = x + y(1-x^2-y^2) + \delta f_2(t,x,y) \ . \end{cases} \tag{12}$$

The main objective of the present paper is the introduction of a notion of Morse-Smale process which allows the comparison as semiflows of perturbed with unperturbed processes.

Proposition 1. *Any C^k evolution process $S_f(\cdot, \cdot)$ on $\mathcal{B}$ induces a C^k semiflow $\mathcal{T}_f(\cdot)$ on $E = \mathbb{R} \times \mathcal{B}$.*

Proof. In fact, for any $s \geq 0$ and $(\tau, x) \in \mathbb{R} \times \mathcal{B}$ one defines

$$\mathcal{T}_f(s)(\tau, x) := (\tau + s, S_f(\tau + s, \tau)x) \tag{13}$$

and it is easy to see that $\mathcal{T}_f$ is a C^k semiflow (see [30, Theorem 2.4] and [2, p. 56]). $\qquad\square$

Corollary 1. *Let $D_1[S_f(t, \tau)x]_{t=\tau}$ be defined as in Definition 3.6. Then, assuming that the domain $\mathcal{D}_\Delta(S)$ has a differentiable structure, the semiflow above induces a vector field X_f defined in $\mathbb{R} \times \mathcal{B}$ by*

$$X_f(\tau, x) = (1, D_1[S_f(\tau + s, \tau)x]_{s=0}) = (1, D_1[S_f(t, \tau)x]_{t=\tau}) \ .$$

Thus, the vector field X_f defines the autonomous ODE in $E = \mathbb{R} \times \mathcal{B}$ of equations

$$\begin{cases} \tau' = 1 \\ x' = D_1[S_f(t, \tau)x]_{t=\tau} \ . \end{cases} \tag{14}$$

By eliminating time, one arrives to the nonautonomous system in $\mathcal{B}$ given by

$$\frac{dx}{d\tau} = D_1[S_f(t, \tau)x]_{t=\tau} := F_f(\tau, x) \ . \tag{15}$$

Remark 2. If $z = z(t)$ is a global solution for $S_f(\cdot, \cdot)$ then the *global curve (trajectory)* $\xi(\tau) = (\tau, z(\tau))$, $\tau \in \mathbb{R}$, is a global solution for $\mathcal{T}_f(\cdot)$. In fact, we have

$$\mathcal{T}_f(s)\xi(\tau) = (\tau + s, S_f(\tau + s, \tau)z(\tau)) = (\tau + s, z(\tau + s)) = \xi(\tau + s), \ s \geq 0, \ \tau \in \mathbb{R} \ .$$

5. **Pullback attractors.** A central object of the study of any dynamical system is the asymptotic behavior of its solutions. This leads to the consideration of a global attractor which under many forms is a notion pervasive in the literature. Hereafter we will use the following notion of pullback attractor, see [30, 8] and the references therein.

Definition 5.1. A family of nonempty compact sets $\{A_f(t) \subset \mathcal{B}, t \in \mathbb{R}\}$ is a *pullback attractor* of the evolution process $S_f(\cdot, \cdot)$ if it is invariant under $S_f(\cdot, \cdot)$, attracts all bounded subsets B of $\mathcal{B}$ in the *pullback sense*, that is

$$\lim_{\tau \to -\infty} \text{dist}(S_f(t, \tau)B, A_f(t)) = 0 \text{ for all } t \in \mathbb{R} \ ,$$

where $\text{dist}(\cdot, \cdot)$ represents the Hausdorff semidistance, and is the minimal family of closed sets pullback attracting all bounded sets of $\mathcal{B}$.

For our purposes it is essential to consider the characterization of pullback attractors for evolution processes and their behavior under perturbation. These topics have been considered in the literature and here, for completion, we only restate the needed results. The next Proposition follows essentially from [8, Theorem 1.17, Corollary 1.18 and Lemma 1.19].

Proposition 2. *If a pullback attractor $\{A_f(t) \subset \mathcal{B}, t \in \mathbb{R}\}$ does exist for an evolution process $S_f(\cdot, \cdot)$ on $\mathcal{B}$ and $\bigcup_{t \in \mathbb{R}} A_f(t)$ is bounded in $\mathcal{B}$, then, for each $t \in \mathbb{R}$, we have that $A_f(t)$ is the set of all values $z(t)$ where $z: \mathbb{R} \to \mathcal{B}$ is a global and bounded solution for $S_f(\cdot, \cdot)$. Moreover, if $S_f(\cdot, \cdot)$ is autonomous, then each $A_f(t)$ of the family coincides with the global attractor A_f of the associated semiflow.*

To study the dependence of global or pullback attractors on the parameter $f \in \mathcal{F}$ the appropriate notion is upper semicontinuity. The following proposition is motivated by [30, Theorem 3.36]. See also [8, Proposition 1.20 and Theorem 3.6].

Proposition 3. *Suppose that the global evolution process* $S \colon \mathcal{F} \times \Delta_+ \times \mathcal{B} \to \mathcal{B}$ *has a pullback attractor* $\{A_f(t) \subset \mathcal{B}, \ t \in \mathbb{R}\}$ *for each* $f \in \mathcal{F}$. *If the closure of* $\bigcup_{f \in \mathcal{F}} \bigcup_{t \in \mathbb{R}} A_f(t)$ *is compact, then the map* $\mathcal{F} \ni f \mapsto A_f(t)$ *is upper semicontinuous for all* $t \in \mathbb{R}$, *that is*

$$\lim_{f \to f_0 \in \mathcal{F}} \mathrm{dist}(A_f(t), A_{f_0}(t)) = 0 \ \text{for all } t \in \mathbb{R} \ .$$

6. **Hyperbolic and partially hyperbolic solutions.** Consider an evolution process $S_f(t, \tau)$ as in Definition 3.2 and let $F_f(\tau, x)$ be defined as in (15).

Definition 6.1. A global and bounded solution $z_0 = z_0(t)$ of

$$\frac{dx}{d\tau} = F_f(\tau, x) \tag{16}$$

is said to be *hyperbolic* if the linear equation

$$y' = B_f(\tau)y \ , \tag{17}$$

with $B_f(\tau) := \frac{\partial F_f}{\partial x}(\tau, z_0(\tau))$, possesses an *exponential dichotomy*, see [25, Definition 7.6.1]. In this case, if $F_f(\tau, x)$ is of class C^1 and under mild conditions (see [30, Section 6.2], [2, Section 6.2], [25, Chapter 6], [14] and [5]), then z_0 has a positively invariant *local stable C^1 manifold* and a negatively invariant *local unstable C^1 manifold*, the invariance being under the semiflow $\mathcal{T}_f$ on $E = \mathbb{R} \times \mathcal{B}$, and both manifolds contain the graph $\alpha = \{(\tau, z_0(\tau)) \colon \tau \in \mathbb{R}\}$ of z_0; one denotes them by $W_{loc}^s(\alpha)$ and $W_{loc}^u(\alpha)$, respectively.

We assume in the following that any evolution process $S_f(\cdot, \cdot)$ considered admits a pullback attractor satisfying the hypotheses of Corollary 1, Propositions 2 and 3. In addition, we suppose that $S_f(\cdot, \cdot)$ has the following *reversibility property*: $S_f(t, \tau)$ restricted to $A_f(\tau)$ is injective onto $A_f(t)$ for all $t \geq \tau \in \mathbb{R}$ and all $f \in \mathcal{F}$, (hence a homeomorphism); also the derivative of $S_f(t, \tau)$ restricted to $A_f(\tau)$ is injective.

Definition 6.2. Applying the semiflow $\mathcal{T}_f$ to $W_{loc}^u(\alpha)$ one obtains the *(global) unstable manifold* $W^u(\alpha)$ of z_0 which is, in general, immersed in $\mathbb{R} \times \mathcal{B}$.

If instead, the linear equation (17) corresponding to a global and bounded solution z_0 of (16) admits an *exponential trichotomy* (see [2, Section 8.1]), the solution z_0 is said to be *partially hyperbolic*. In this case, under adequate conditions on $F_f(\tau, x)$, z_0 possesses a *center manifold* invariant under the semiflow $\mathcal{T}_f$, see [2, Chapter 8], [40, 12]. Here we will consider isolated invariant manifolds that are *normally hyperbolic* in the sense that the linearization around each solution with trajectory in the manifold has an exponential trichotomy with a two dimensional center (or *intermediate*, [2, p. 172]) manifold.

Remark that all these manifolds in $\mathbb{R} \times \mathcal{B}$ have dimension at least 1 since they must contain the graph $\alpha = \{(t, z_0(t)) : t \in \mathbb{R}\}$ of the trajectory corresponding to the solution z_0. Hence, hyperbolicity of the solution z_0 corresponds to a center manifold of dimension $r = 1$. A periodic orbit with minimum period $T > 0$ of a semiflow T_f is hyperbolic if the corresponding Poincaré map has exactly one simple multiplier $\mu = 1$. In this case, the solution z_0 of the autonomous evolution process $S_f(t, \tau) = T_f(t - \tau)$, $t \geq \tau$, is partially hyperbolic. The hyperbolic periodic

orbit of T_f generates a *cylinder* in $\mathbb{R} \times \mathcal{B}$ composed of graphs of the solutions $z_\tau(\cdot) = T_f(\tau)z_0(\cdot), 0 \leq \tau < T$, hence defining a center manifold of dimension $r = 2$ embedded in $\mathbb{R} \times \mathcal{B}$ which is normally hyperbolic.

7. **Isolated invariant manifolds.** In the following we collect some results on the behavior of certain isolated invariant manifolds for differential equations under nonautonomous perturbations. We consider specifically invariant manifolds associated to hyperbolic equilibria and hyperbolic periodic orbits.

7.1. **Hyperbolic equilibria.** Let $Y \subseteq X$ be Banach spaces. Assume L is a linear operator generating a C_0-semigroup of bounded linear operators in X and let $g \in C^1(Y, X)$. We consider either the finite dimensional case $Y = X = \mathbb{R}^n$ or, in infinite dimensions, the case of $-L$ sectorial and $Y = X^\alpha$, the fractional power space of $-L$ with appropriate $\alpha \in (0, 1)$. We denote by $C_\varepsilon(\mathbb{R}, C^1(Y, X))$ the metric space of continuous functions $f = f(t, z)$ uniformly Lipschitz continuous for $t \in \mathbb{R}$ and differentiable in z with uniform ε-bounded C^1-norm, i.e.

$$\sup_{t \in \mathbb{R}} \sup_{z \in Y} \left(\|f(t, z)\|_X + \|f_z(t, z)\|_{\mathcal{L}(Y, X)} \right) \leq \varepsilon .$$

Let z_0 denote a hyperbolic equilibrium of the autonomous system

$$\dot{z} = Lz + g(z) . \tag{18}$$

Then for $f_\varepsilon \in C_\varepsilon(\mathbb{R}, C^1(Y, X))$ the nonautonomous perturbed equation

$$\dot{z} = Lz + g(z) + f_\varepsilon(t, z) \tag{19}$$

has the following properties:

There is an $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$ the nonautonomous system (19) has a global solution $z_\varepsilon = z_\varepsilon(t)$ uniformly close to z_0. This follows from the fixed point property of the variation of constants equation associated to (19) using the hyperbolicity of z_0 and the projections onto its stable and unstable manifolds.

Moreover, $z_\varepsilon(\cdot) \to z_0$ as $\varepsilon \to 0$ and $z_\varepsilon(\cdot)$ is a hyperbolic solution of (19) since the linearization of (19) around z_ε has an exponential dichotomy. This follows from [25, Theorem 7.6.11], see also [37, Theorem 76.1]. For more details see [5].

Remark that the trajectory (t, z_0) corresponds to the hyperbolic solution of $(19)_{\varepsilon=0}$ and the trajectory $(t, z_\varepsilon(t))$ is the corresponding perturbed *isolated invariant manifold*.

According to Definition 3.2 and Remark 1 equation (19) defines an evolution process on $\mathcal{B} = Y$ with $\mathcal{F} = C_{\varepsilon_0}(\mathbb{R}, C^1(Y, X))$.

7.2. **Hyperbolic periodic orbits.** Let $\Gamma = \{\xi_0(t), 0 \leq t < \omega\} \subset Y$ denote a hyperbolic periodic orbit of (18) with $\xi_0(\cdot)$ a periodic solution with period $\omega > 0$. The corresponding cylinder of trajectories

$$\mathcal{M}_0 = \{(t, \xi_0(\tau + t)), 0 \leq \tau < \omega, t \in \mathbb{R}\}$$

is an isolated invariant manifold of (18) in (t, z)-space with parameters $(t, \tau) \in \mathbb{R} \times [0, \omega)$.

Then for $f_\varepsilon \in C_\varepsilon(\mathbb{R}, C^1(Y, X))$ the nonautonomous perturbed equation (19) satisfies the following:

There is an $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$ the nonautonomous system (19) has an *isolated invariant manifold* $\mathcal{M}_\varepsilon$ uniformly close to $\mathcal{M}_0$ of the form

$$\mathcal{M}_\varepsilon = \{(t, \xi_\varepsilon(t, \tau)), 0 \leq \tau < \omega, t \in \mathbb{R}\} .$$

Moreover $\mathcal{M}_\varepsilon \to \mathcal{M}_0$ as $\varepsilon \to 0$ and is normally hyperbolic since the linearization of (19) around each solution $\xi_\varepsilon(\cdot, \tau)$ with trajectory in $\mathcal{M}_\varepsilon$ has an exponential trichotomy with a two dimensional center manifold.

Remark that, in general, as $\varepsilon \to 0$ the solution $\xi_\varepsilon(\cdot, 0)$ does not approach $\xi_0(\cdot)$ as shown by the examples of nonautonomous perturbations of autonomous ODEs, see system (1) with the tangential perturbation (7).

The persistence of these isolated invariant manifolds is supported by the following results:

For $z \in \mathbb{R}^n$, by [22, 39] there is a local change of coordinates around $\xi_0(\cdot)$ of the form

$$z = \xi_0(\theta) + Q(\theta)\rho ,$$

with $(\theta, \rho) \in \mathbb{R} \times \mathbb{R}^{n-1}$ which applied to (19) yields

$$\begin{cases} \dot{\theta} = 1 + S(\theta, \rho) + S_\varepsilon(t, \theta, \rho) \\ \dot{\rho} = A(\theta)\rho + R(\theta, \rho) + R_\varepsilon(t, \theta, \rho) \end{cases}$$

where $S(\cdot, \rho) = O(|\rho|)$, $R(\cdot, \rho) = O(|\rho|^2)$, $(S_\varepsilon, R_\varepsilon) \in C_\varepsilon(\mathbb{R}, C^1(\mathbb{R} \times \mathbb{R}^{n-1}))$, and $A(\cdot)$ is periodic with period ω. By Floquet theory a fundamental system of solutions of the linear equation $d\rho/d\theta = A(\theta)\rho$ has the form $P(\theta)e^{B\theta}$ with $P(\cdot)$ periodic with period ω, and the change of coordinates $\rho(t) = P(\theta)r(t)$ yields

$$\begin{cases} \dot{\theta} = 1 + S^F(\theta, r) + S_\varepsilon^F(t, \theta, r) \\ \dot{r} = Br + R^F(\theta, r) + R_{0,\varepsilon}^F(t, \theta, r) + R_\varepsilon^F(t, \theta, r) \end{cases} \tag{20}$$

where $S^F(\cdot, r) = O(|r|)$, $R^F(\cdot, r) = O(|r|^2)$, $(S_\varepsilon^F, R_\varepsilon^F) \in C_\varepsilon(\mathbb{R}, C^1(\mathbb{R} \times \mathbb{R}^{n-1}))$ and $R_{0,\varepsilon}^F(\cdot, \cdot, r) = O(\varepsilon|r|)$, see [22, Chapter VII.1]. This brings about the role of the *characteristic multipliers* of the hyperbolic periodic orbit of $\xi_0(\cdot)$. If $\lambda_j, j = 1, \ldots, n-1$, denote the eigenvalues of B, then the characteristic multipliers are given by

$$\mu_j = e^{\omega\lambda_j} , \ 1 \le j \le n-1 .$$

Hence system (20) is equivalent to

$$\begin{cases} \dot{\theta} = 1 + \tilde{S}^F(\theta, r_1, r_2) + \tilde{S}_\varepsilon^F(t, \theta, r_1, r_2) \\ \dot{r}_1 = B_1 r_1 + \tilde{R}^F(\theta, r_1, r_2) + \tilde{R}_{0,\varepsilon}^F(t, \theta, r_1, r_2) + \tilde{R}_\varepsilon^F(t, \theta, r_1, r_2) \\ \dot{r}_2 = B_2 r_2 + \hat{R}^F(\theta, r_1, r_2) + \hat{R}_{0,\varepsilon}^F(t, \theta, r_1, r_2) + \hat{R}_\varepsilon^F(t, \theta, r_1, r_2) \end{cases} \tag{21}$$

where, by the hyperbolicity of $\xi_0(\cdot)$, there is a real $\alpha_0 > 0$ such that the matrices B_1 and B_2 satisfy

$$\mathrm{Re}\{\sigma(B_1)\} \ge \alpha_0 , \ \mathrm{Re}\{\sigma(B_2)\} \le -\alpha_0 .$$

Then, by [22, Theorem VII.2.1], this system has an isolated invariant manifold of the form

$$\widetilde{\mathcal{M}}_\varepsilon = \{(t, \theta, r_1, r_2) : r_1 = G_1(t, \theta, \varepsilon), r_2 = G_2(t, \theta, \varepsilon), 0 \le \theta < \omega, t \in \mathbb{R}\} ,$$

with G_1, G_2, continuous of class C^1 in θ and satisfying $G_1(t, \theta, 0) = G_2(t, \theta, 0) = 0$. Moreover, $\widetilde{\mathcal{M}}_\varepsilon$ has a *saddle-point structure* in the sense that any solution of (21) is such that $r_1 \to 0$ exponentially as $t \to -\infty$ and $r_2 \to 0$ exponentially as $t \to \infty$ (both with asymptotic phase), see [22, Theorem VII.7.1].

This shows the existence of the isolated invariant manifold $\mathcal{M}_\varepsilon$ of (19) with

$$\xi_\varepsilon(t, \theta) = \xi_0(\theta) + Q(\theta)F(t, \theta, \varepsilon) ,$$

contained in a tubular neighborhood of the cylinder $\mathcal{M}_0$. The exponential trichotomy of each solution $\xi_\varepsilon(\cdot, \theta)$ follows from the saddle-point property of $\widetilde{\mathcal{M}_\varepsilon}$. We also observe that the flow of (19) on $\mathcal{M}_\varepsilon$ is equivalent to the equation

$$\frac{d\theta}{dt} = 1 + S_\varepsilon(t, \theta, 0) \ ,$$

with $\theta(0) \in [0, \omega)$, see [21, Theorem 16–1].

For $z \in X^\alpha$, by [25, Section 9.2] there is a local change of coordinates around $\xi_0(\cdot)$ generalizing the result of [22, 39]. In fact, by [25, Theorem 9.2.2] there exists a closed subspace $Y_0 \subset X$ of codimension 1 and a C^2 map $Q : \Gamma \to \mathcal{L}(X)$ such that for each $y \in \Gamma$ the linear map $Q(y)$ is an isomorphism of Y_0 onto its image with $Q(y)Y_0 \oplus T_y\Gamma = X$, and a neighborhood of Γ has the form

$$N_\Gamma = \{y + Q(y)x : y \in \Gamma, x \in Y_0, \|x\| < \delta\} \ ,$$

with $(y, x) \mapsto y + Q(y)x$ a C^2 diffeomorphism onto this neighborhood.

Then, assuming $t \mapsto \xi_0(t) \in X$ is C^2 and using $y = \xi_0(s)$, we introduce in N_Γ the coordinates (s, x) and write

$$z = \xi_0(s) + \bar{Q}(s)x \ , \ (s, x) \in \mathbb{R} \times Y_0 \ ,$$

for $\|x\| < \delta$, where $\bar{Q}(s) = Q(\xi_0(s))$ and $s \mapsto \bar{Q}(s)$ is of class C^2 and periodic with period $\omega > 0$.

Using these coordinates, the flow of (18) in the neighborhood N_Γ has the form

$$\begin{cases} \dot{s} = 1 + S(s, x) \\ \dot{x} = Bx + B_0(s)x + R(s, x) \ , \end{cases} \tag{22}$$

where the linear operator B with $D(B) = D(L) \cap Y_0$ is sectorial in Y_0, and $B_0(s) \in \mathcal{L}(Y_0)$ is a family of bounded linear operators ω-periodic in s. Moreover, defining $Y_0^\alpha = Y_0 \cap X^\alpha$, we have that S and R are C^1 functions of $s \in \mathbb{R}$ and $x \in Y_0^\alpha$, $\|x\|_\alpha < \bar{\delta}$, which are also ω-periodic in s and vanish for $x = 0$, see [25, Section 9.2]. In fact, by an appropriate choice of projection $P \in \mathcal{L}(X)$, with $N(P) \subset Y_0$, derived from an extension of Whitney's embedding theorem (see [25, Theorem 9.2.1]), we have $B = (I - P)L|Y_0$, which implies that B is sectorial. Similarly, the flow of (19) in the neighborhood N_Γ becomes

$$\begin{cases} \dot{s} = 1 + S(t, s, x) \\ \dot{x} = Bx + B_0(s)x + R(t, s, x) \ , \end{cases} \tag{23}$$

with C^1 functions $S(t, s, x)$ and $R(t, s, x)$ of the form

$$S(t, s, x) = S(s, x) + S_\varepsilon(t, s, x)$$
$$R(t, s, x) = R(s, x) + R_\varepsilon(t, s, x) \ ,$$

where $S(\cdot, x) = O(\|x\|_\alpha)$, $R(\cdot, x) = O(\|x\|_\alpha^2)$ and $(S_\varepsilon, R_\varepsilon) \in C_\varepsilon(\mathbb{R}, C^1(\mathbb{R} \times Y_0^\alpha, \mathbb{R} \times Y_0))$.

Let $T(t, \tau), t \geq \tau$, denote the family of evolution operators on X generated by the linearization of (18) around $\xi_0(t)$, that is $T(t, \tau)z_0 = z(t, \tau, z_0)$ for $z_0 \in X$ where $z(t) = z(t, \tau, z_0), t \geq \tau$, solves

$$\dot{z} = Lz + g_z(\xi_0(t))z \ , \ z(\tau) = z_0 \ .$$

The *characteristic multipliers* of $\xi_0(t)$ are the nonzero eigenvalues of the *Poincaré map*

$$U(t) = T(t + \omega, t) \ .$$

We recall that the eigenvalues of the Poincaré map are independent of t and if L has compact resolvent then $\sigma(U(\cdot)) \setminus \{0\}$ is composed entirely of eigenvalues $\mu_j, j = 1, 2, \ldots,$ [25, Lemma 7.2.2].

Due to hyperbolicity, $\mu = 1$ is a simple characteristic multiplier of $\xi_0(t)$, with the remaining spectrum of $U(\cdot)$ away from the unit circle. Moreover, the multipliers $\neq 1$ are exactly the multipliers of the linear equation $\frac{dx}{ds} = Bx + B_0(s)x$ corresponding to the linearization around $x = 0$ of the second equation in (22). This equation has no multipliers on the unit circle and, therefore, has an exponential dichotomy. See [25, Section 9.3] for details.

Let $\widetilde{T}(\cdot, \cdot)$ denote the linear evolution process corresponding to $\frac{dx}{ds} = Bx + B_0(s)x$ and consider the Poincaré map $\widetilde{U}(s) = \widetilde{T}(s + \omega, s)$. By [25, Theorem 7.2.3], if σ_1 is a spectral set of $\sigma(\widetilde{U}(\cdot)) \setminus \{0\}$ then, for each s the space Y_0 decomposes as $Y_0 = X_1(s) \oplus X_2(s)$, the direct sum of closed invariant subspaces under $\widetilde{U}(s)$, and

$$\sigma(\widetilde{U}(s)|X_1(s)) = \sigma_1 \quad , \quad \sigma(\widetilde{U}(s)|X_2(s)) = \sigma(\widetilde{U}(s)) \setminus \sigma_1 \ ,$$

where $\widetilde{T}(s_2, s_1), s_2 \geq s_1$, maps $X_1(s_1)$ one-to-one onto $X_1(s_2)$. Moreover, there is a family of bounded invertible linear operators $P_1(s) : X_1(s_0) \to X_1(s)$, with period ω in s and $P_1(s_0) = \mathrm{Id}$, and a bounded linear operator C on $X_1(s_0)$ with $e^{\omega \sigma(C)} = \sigma_1$ such that

$$\widetilde{T}(s_2, s_1)x = P_1(s_2)e^{C(s_2 - s_1)}P_1^{-1}(s_1)x \ ,$$

for all $x \in X_1(s_1)$ and all $s_2 \geq s_1$. Then, using a decomposition $x = x_1 + x_2 \in X_1(s) \oplus X_2(s)$ and letting $x_1 = P_1(s)y$ with $y \in X_1(s_0)$, the above system (23) is equivalent to

$$\begin{cases} \dot{s} = 1 + S^F(t, s, y, x_2) \\ \dot{y} = Cy + \tilde{R}^F(t, s, y, x_2) \\ \dot{x}_2 = D(s)x_2 + \hat{R}^F(t, s, y, x_2) \ , \end{cases} \quad (24)$$

where $D(s)$ is a sectorial operator in $X_2(s)$, see [25, Section 7.2 ex.2]. This again brings about the role of the characteristic multipliers in the discussion of the asymptotic behavior of the flow in the neighborhood N_Γ.

Then, by a direct application of [25, Theorem 9.1.1] system (23) has an isolated invariant manifold of the form

$$\widetilde{\mathcal{M}}_\varepsilon = \{(t, s, x) : x = G(t, s, \varepsilon), 0 \leq s < \omega, t \in \mathbb{R}\} \ ,$$

with continuous G of class C^1 in s and $G(t, s, 0) = 0$ as required. Moreover, the saddle-point nature of this manifold is exposed in (24) by choosing the spectral set σ_1 as the set of all multipliers outside the unit circle. It also exhibits the exponential trichotomy of each global solution $\xi_\varepsilon(\cdot, s)$ of (19) with trajectory in $\mathcal{M}_\varepsilon$.

8. **Recurrent behavior.** The definition of Morse-Smale systems on $\mathcal{B}$ involves the concept of nonwandering behavior for hyperbolic critical orbits, see [34, 24, 32]. For the case of processes, we instead introduce the notion of recurrent behavior on $\mathbb{R} \times \mathcal{B}$ neighborhoods of graphs of global solutions (trajectories).

Let $B(l, 0)$ be the ball in $\mathcal{B}$ of radius $l > 0$ centered at the origin and let $\xi(t) = (t, z(t))$ be the trajectory in $\mathbb{R} \times \mathcal{B}$ corresponding to a bounded global solution z in $\mathcal{B}$. When the process has a pullback attractor, one defines the neighborhood $U(t, l, z(t))$ of $z(t)$ in $A_f(t)$ by the expression

$$U(t, l, z(t)) := \{z(t) + B(l, 0)\} \cap A_f(t) \quad , \quad t \in \mathbb{R} \ .$$

Definition 8.1. A trajectory $\xi \colon \mathbb{R} \to \mathbb{R} \times \mathcal{B}$ associated to a global bounded solution $z(\cdot)$ of (16) given by $\xi(t) := (t, z(t))$, $t \in \mathbb{R}$, is said to produce a *recurrent behavior* of $S_f(\cdot, \cdot)$ (or of $T_f(\cdot)$) if given a time $\bar{t} \in \mathbb{R}$ and a neighborhood $U(\bar{t}, l, z(\bar{t}))$ of $z(\bar{t})$ in $A_f(\bar{t}) \subset \mathcal{B}$ there exists $\bar{z} \in U(\bar{t}, l, z(\bar{t}))$ with $\bar{z} \neq z(\bar{t})$ such that for any $\overline{T} > 0$ the solution $\tilde{z}(\cdot)$ of (16) with $\tilde{z}(\bar{t}) = \bar{z}$ satisfies $\tilde{z}(t_{\pm}) \in U(t_{\pm}, l, z(t_{\pm}))$ for some $t_+ > \bar{t} + \overline{T}$ and $t_- < \bar{t} - \overline{T}$. For simplicity, in this case we say that the trajectory $\xi \colon \mathbb{R} \to \mathbb{R} \times \mathcal{B}$ is recurrent.

This definition of recurrent behavior is illustrated in Figure 2, where $\tilde{\xi} := (t, \tilde{z}(t))$ and $U_\tau := U(\tau, l, z(\tau))$.
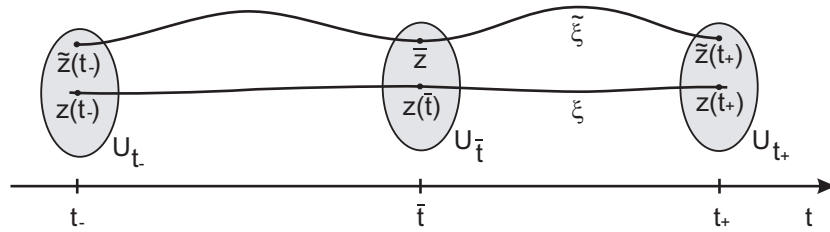

FIGURE 2. Recurrent behavior for $\xi = (t, z(t))$. Here $t_+ - \bar{t} > \overline{T}$ and $\bar{t} - t_- > \overline{T}$.

Definition 8.2. Consider a global bounded solution $z(\cdot)$ of (16) and let $\mathcal{M}_z$ denote a manifold containing the trajectory $\xi(t) = (t, z(t))$, $t \in \mathbb{R}$, such that $\mathcal{M}_z$ is invariant under the semiflow $\mathcal{T}_f$ on $\mathbb{R} \times \mathcal{B}$ and has t-section $\mathcal{M}_z^t := \mathcal{M}_z \cap (\{t\} \times \mathcal{B})$ with natural $\mathcal{B}$-projection $P_\mathcal{B} \mathcal{M}_z^t$ contained in $A_f(t)$. The trajectory ξ is said to produce a *recurrent behavior of $S_f(\cdot, \cdot)$ in the complement of $\mathcal{M}_z$* if given a time $\bar{t} \in \mathbb{R}$ and a neighborhood $U(\bar{t}, l, z(\bar{t}))$ of $z(\bar{t})$ in $A_f(\bar{t}) \subset \mathcal{B}$ there exists $\bar{z} \in A_f(\bar{t}) \setminus P_\mathcal{B} \mathcal{M}_z^{\bar{t}}$ (hence $\bar{z} \neq z(\bar{t})$) such that for any $\overline{T} > 0$ the solution $\tilde{z}(\cdot)$ of (16) with $\tilde{z}(\bar{t}) = \bar{z}$ satisfies $\tilde{z}(t_{\pm}) \in U(t_{\pm}, l, z(t_{\pm}))$ for some $t_+ > \bar{t} + \overline{T}$ and $t_- < \bar{t} - \overline{T}$. For simplicity, in this case we say that the trajectory $\xi \colon \mathbb{R} \to \mathbb{R} \times \mathcal{B}$ is *recurrent outside $\mathcal{M}_z$*.

Remark 3. A trajectory in $\mathbb{R} \times \mathbb{R}^n$ of an autonomous evolution process corresponding to a constant hyperbolic solution in $\mathbb{R}^n$ with a homoclinic orbit is recurrent. On the other hand, any solution of an autonomous evolution process corresponding to a constant solution of a gradient Morse-Smale flow in $\mathbb{R}^n$ defines a nonrecurrent trajectory in $\mathbb{R} \times \mathbb{R}^n$.

Remark 4. In contrast with the second observation of the previous remark, a trajectory in $\mathbb{R} \times \mathbb{R}^n$ corresponding to a periodic solution in $\mathbb{R}^n$ with a hyperbolic orbit is always recurrent. In fact, the process exhibits recurrent behavior on the invariant cylinder generated by the periodic orbit, which is the center manifold in $\mathbb{R} \times \mathbb{R}^n$ of the partially hyperbolic periodic solution.

Example 1. A particular case of that special situation is given by the autonomous Morse-Smale ODE system on $\mathcal{B} = \mathbb{R}^2$, considered in (3) of Section 2:

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \, . \end{cases} \tag{25}$$

The cylinder in this case is the subset of $\mathbb{R} \times \mathbb{R}^2$ given by $\mathbb{R} \times \{(x,y) : x^2 + y^2 = 1\}$ which is an invariant manifold for the lifted system:

$$\begin{cases} t' = 1 \\ x' = -y + x(1 - x^2 - y^2) \\ y' = x + y(1 - x^2 - y^2) \ . \end{cases} \quad (26)$$

Observe that all the global solutions of (26) corresponding to the periodic solutions of (25) are partially hyperbolic and their graphs are fibers of the same normally hyperbolic cylinder, i.e., the union of all graphs of trajectories $(t, z_\tau(t))$ corresponding to the periodic solutions $z_\tau(t) = (\cos(t + \tau), \sin(t + \tau))$ in $\mathbb{R}^2$ for $\tau \in [0, 2\pi)$. Moreover, the recurrent behavior is restricted to this cylinder. In fact, any global solution $(t, z(t))$ of the lifted system (26) different from the trivial solution $(t, 0, 0)$ and not in the cylinder is in the stable manifold of a solution corresponding to a periodic orbit, and $z(t)$ either goes to $(0, 0)$ or becomes unbounded as $t \to -\infty$. Therefore, if $(t, z(t))$ denotes a normally hyperbolic solution on the cylinder, the process generated by (25) when restricted to the complement of the cylinder has no recurrent behavior, although $(t, z(t))$ produces recurrent behavior on the cylinder. Moreover, this recurrent behavior on the cylinder is mostly preserved by the tangential components of generic perturbations, as illustrated in Section 2. This suggests that in the case of solutions on normally hyperbolic manifolds we should only study the (non)occurrence of recurrent behavior outside the center manifolds, since on the center manifolds recurrent behavior is always expected.

9. **Morse-Smale evolution processes.** According to Definition 3.3 the evolution process $S_f(\cdot, \cdot)$ depends continuously on $f \in \mathcal{F}$ where the parameter space $\mathcal{F}$ is a metric space appropriately chosen depending on each problem. Recall that for each $f \in \mathcal{F}$ the evolution process $S_f(\cdot, \cdot)$ is assumed to:

(i) admit a pullback attractor satisfying the hypotheses of Propositions 2, 3;

(ii) have the reversibility property, (see Section 6).

Our definition of Morse-Smale process follows from the previous observations regarding the occurrence and location of recurrent behavior in Example 1. We start by identifying the sets of global solutions which are relevant for this definition.

We let $\mathcal{M}_f^h \subset \mathbb{R} \times \mathcal{B}$ denote the set of all graphs of hyperbolic solutions of $S_f(\cdot, \cdot)$,

$$\mathcal{M}_f^h = \{(t, z(t)) : z(\cdot) \text{ hyperbolic solution of } S_f(\cdot, \cdot), t \in \mathbb{R}\} \ .$$

We also let $\mathcal{M}_f^{nh}$ denote the set of all normally hyperbolic isolated invariant manifolds $\mathcal{M}_f^T$ satisfying the following:

(T1) $\mathcal{M}_f^T$ are fibered by graphs of partially hyperbolic solutions of $S_f(\cdot, \cdot)$ possessing exponential trichotomy with two dimensional center manifold;

(T2) for each $t \in \mathbb{R}$ the natural $\mathcal{B}$-projection of the t-section of $\mathcal{M}_f^T$ (which is in $A_f(t)$) is homeomorphic to a circle.

Such invariant manifolds will be called *tubes* hereafter.

For each tube $\mathcal{M}_f^T$ we define its *unstable manifold*, $W^u(\mathcal{M}_f^T)$, as the union of all the unstable manifolds of its fibers $\alpha \in \mathcal{M}_f^T$,

$$W^u(\mathcal{M}_f^T) = \bigcup_{\alpha \in \mathcal{M}_f^T} W^u(\alpha) \ .$$

Similarly we define the *local stable manifold*, $W_{loc}^s(\mathcal{M}_f^T)$.

We say that two manifolds $W^u(\mathcal{M}_{f,1}^T)$ and $W_{loc}^s(\mathcal{M}_{f,2}^T)$ are *transverse* if for all $t \in \mathbb{R}$ their t-sections have $\mathcal{B}$-projections transversal in $\mathcal{B}$ in the usual sense. Likewise we define transversality between unstable and local stable manifolds of hyperbolic solutions, $W^u(\alpha_1), W_{loc}^s(\alpha_2)$, or hyperbolic solutions and tubes.

Definition 9.1. Assume that:
1. The sets $\mathcal{M}_f^h$ and $\mathcal{M}_f^{nh}$ are finite;
2. All unstable manifolds of elements in $\mathcal{M}_f^h$ and $\mathcal{M}_f^{nh}$ are finite dimensional. Moreover, all intersections between these unstable manifolds and local stable manifolds are transverse;
3. All trajectories of hyperbolic solutions (with graphs in $\mathcal{M}_f^h$) are nonrecurrent. All trajectories of partially hyperbolic solutions with graphs in tubes (elements of $\mathcal{M}_f^{nh}$) are nonrecurrent outside the tubes. Moreover, the sets $A_f(t)$ in the pullback attractor of $S_f(\cdot, \cdot)$ are given by $A_f(t) = P_{\mathcal{B}}\mathcal{A}_f^t$, the natural $\mathcal{B}$-projections of the t-sections of $\mathcal{A}_f$ defined as the union of all the unstable manifolds of elements in $\mathcal{M}_f^h$ and $\mathcal{M}_f^{nh}$, $\mathcal{A}_f = \bigcup_{\alpha \in \mathcal{M}_f^h \cup \mathcal{M}_f^T} W^u(\alpha)$.

Under these three conditions we say that the parameter $f \in \mathcal{F}$ defines a Morse-Smale evolution process on $\mathcal{B}$.

We next show the following:

Theorem 9.2. *If $T_f(t), t \geq 0$, is a Morse-Smale semiflow of C^k transformations of $\mathcal{B}$ (with or without hyperbolic periodic orbits) then the autonomous C^k evolution process on $\mathcal{B}$ defined by $S_f(t, \tau) = T_f(t - \tau)$ is a Morse-Smale evolution process on $\mathcal{B}$.*

Proof. If A_f denotes the global attractor of the semiflow T_f then the family of compact sets $\{C_f(t) = A_f, t \in \mathbb{R}\}$ is the pullback attractor of the autonomous evolution process defined by $S_f(t, \tau) = T_f(t - \tau)$. In fact the family $\{C_f(t) = A_f, t \in \mathbb{R}\}$ attracts all bounded subsets $B \subset \mathcal{B}$ in the pullback sense,

$$\lim_{\tau \to -\infty} \text{dist}(S_f(t, \tau)B, A_f) = \lim_{\tau \to -\infty} \text{dist}(T_f(t - \tau)B, A_f) = 0 \,.$$

The critical elements of T_f are the unit sets $\{z_0\}$ where z_0 is a fixed point, and the periodic orbits. Since T_f is a Morse-Smale semiflow, the set of its critical elements is finite and the fixed points and periodic orbits are hyperbolic. Moreover, their union coincides with the nonwandering set $\Omega(f)$, see [32, Definition 3.1],[24, Definition 6.2.9]. Clearly the solutions $z(t) = z_0$ of $S_f(\cdot, \cdot)$ corresponding to fixed points of T_f are hyperbolic. Hence their graphs $\alpha = \{(\tau, z_0), \tau \in \mathbb{R}\}$ are elements of $\mathcal{M}_f^h$. Also the cylinders generated by the periodic orbits of T_f are normally hyperbolic, see Section 6. Therefore each periodic orbit corresponds to a tube in $\mathcal{M}_f^{nh}$. We next show that $\mathcal{M}_f^h$ and $\mathcal{M}_f^{nh}$ consist exactly of these elements.

Starting with $\mathcal{M}_f^h$ we will argue by contradiction. Let $z(\cdot)$ denote a hyperbolic solution of $S_f(\cdot, \cdot)$, that is, with graph $\alpha = \{(t, z(t)), t \in \mathbb{R}\} \in \mathcal{M}_f^h$. Then, since the evolution process is autonomous, $z(\cdot)$ is a solution of the semiflow T_f with global bounded orbit γ in A_f. By the Morse-Smale property of the semiflow T_f which asserts that the nonwandering set $\Omega(f)$ is the union of the critical elements of T_f, we have that the α-limit and ω-limit sets of the orbit γ are critical elements of the semiflow T_f. We remark that since the periodic solutions of $S_f(\cdot, \cdot)$ are partially hyperbolic, $z(\cdot)$ cannot correspond to a hyperbolic periodic orbit of T_f. Then if $z(\cdot)$ does not correspond to a fixed point its orbit γ must connect two

different critical elements c_1, c_2 of T_f. In fact, for $c_1 = c_2 = c$ we cannot have $W^u(c) \cap W^s_{loc}(c) \supsetneq c$ due to transversality since, in that case, the λ-lemma (see [33, 32, 24]) implies infiniteness of $\Omega(f)$. Next, if c_1 and c_2 are distinct fixed points of T_f, transversality between $W^u(c_1)$ and $W^s_{loc}(c_2)$ also implies the strict inequality $\dim W^u(c_1) > \operatorname{codim} W^s_{loc}(c_2)$ since $W^u(c_1) \cap W^s_{loc}(c_2)$ must contain the positive semi-orbit of γ in $W^s_{loc}(c_2)$. Therefore we have

$$\dim W^u(c_1) > \dim W^u(c_2) . \tag{27}$$

This shows that the linearization of $S_f(\cdot, \cdot)$ around $z(\cdot)$ cannot possess an exponential dichotomy which contradicts the hyperbolicity of $z(\cdot)$. By the same reason c_1 or c_2 cannot be hyperbolic periodic orbits of T_f since the existence of a simple multiplier $\mu = 1$ for the corresponding Poincaré map prevents hyperbolicity of $z(\cdot)$. Therefore we conclude that $z(\cdot)$ corresponds to a fixed point of T_f.

We now show that each tube in $\mathcal{M}^{nh}_f$ is a cylinder generated by a periodic orbit of T_f. Let $\alpha = \{(t, z(t)), t \in \mathbb{R}\}$ denote the graph of a partially hyperbolic solution $z(\cdot)$ of $S_f(\cdot, \cdot)$ corresponding to a fiber of a tube $\mathcal{M}^T_f$ in $\mathcal{M}^{nh}_f$. Repeating the previous arguments we have that $z(\cdot)$ is a solution of the semiflow T_f with global bounded orbit γ in A_f. Moreover the α-limit and ω-limit sets of the orbit γ are critical elements of the semiflow T_f. We remark that since $z(\cdot)$ is partially hyperbolic γ cannot be a hyperbolic fixed point of T_f. Also, the critical elements cannot be fixed points of T_f; in fact, if the α-limit or the ω-limit set of γ is a unit set of a fixed point, the linearization of $S_f(\cdot, \cdot)$ around the solution $z(\cdot)$ cannot admit an exponential trichotomy with a two dimensional center manifold, since the fixed points are hyperbolic. This implies that the critical elements are periodic orbits, C_1 and C_2. Next, arguing again by contradiction, we show that γ cannot connect two different periodic orbits. Assume that the periodic orbit C_1 is the α-limit set of γ, and C_2 is the ω-limit. Consider the Poincaré map for the semiflow T_f defined at a local cross-section of C_1, and let $i(C_1)$ denote the number of multipliers μ_j of this map outside the unit disk, $|\mu_j| > 1$, counting multiplicities. We recall that C_1 is hyperbolic and its unstable manifold is finite dimensional. Then we have $\dim W^u_{loc}(C_1) = i(C_1) + 1$. Likewise, defining $i(C_2)$ for the Poincaré map at a local cross-section of C_2 we have

$$\operatorname{codim} W^s_{loc}(C_2) = i(C_2) . \tag{28}$$

We now consider a point $p \in \gamma$ on the intersection between the global unstable manifold of C_1 and the local stable manifold of C_2, that is $p \in W^u(C_1) \cap W^s_{loc}(C_2)$, and obviously

$$\dim W^u(C_1) = i(C_1) + 1 . \tag{29}$$

These manifolds intersect transversally at the point p and their intersection contains the positive semi-orbit of γ. The same conclusions hold for each of the fibers of the tube $\mathcal{M}^T_f$. Moreover, by property (T2) all the orbits corresponding to fibers of $\mathcal{M}^T_f$ have C_1 as α-limit set and C_2 as ω-limit set. Hence the intersection $W^u(C_1) \cap W^s_{loc}(C_2)$ contains positive semi-orbits of all the solutions corresponding to fibers of $\mathcal{M}^T_f$. This implies that the transversal intersection $W^u(C_1) \cap W^s_{loc}(C_2)$ is at least two dimensional and

$$\operatorname{codim} W^s_{loc}(C_2) \leq \dim W^u(C_1) - 2 .$$

From (28) and (29) we conclude that

$$i(C_2) \leq i(C_1) - 1 .$$

This shows that the linearization of $S_f(\cdot, \cdot)$ around $z(\cdot)$ cannot possess an exponential trichotomy which contradicts the partial hyperbolicity of $z(\cdot)$. Therefore we must have $C_1 = C_2 := C$.

Again by transversality, invoking the λ-lemma and the finiteness of $\Omega(f)$, we cannot have $W^u(C) \cap W^s_{loc}(C) \supsetneq C$. Therefore we conclude that the solution $z(\cdot)$ is a periodic solution of the autonomous semiflow T_f and corresponds to the periodic orbit C. Hence the tube $\mathcal{M}^T_f$ is the cylinder generated by C, and $\mathcal{M}^{nh}_f$ is the set of all cylinders generated by periodic orbits of T_f. This completes the proof of condition (1) of the Morse-Smale Definition 9.1. Condition (2) also follows because, for the autonomous evolution process, we have

$$\mathcal{T}_f(s)(\tau, x) = (\tau + s, T_f(s)x)$$

by Proposition 1. This implies that $\mathcal{B}$-projections of t-sections of $\mathcal{T}_f$-invariant manifolds in $\mathbb{R} \times \mathcal{B}$ are T_f-invariant manifolds in $\mathcal{B}$.

To prove condition (3) of the Morse-Smale Definition 9.1 we first consider the trajectory $\xi(t) = (t, z_0), t \in \mathbb{R}$, corresponding to a hyperbolic fixed point z_0 of T_f in $\mathcal{B}$. For a given time $\bar{t} \in \mathbb{R}$ let $U(\bar{t}, l, z_0)$ denote the neighborhood of z_0 in $A_f(\bar{t}) \subset \mathcal{B}$ with $l > 0$ sufficiently small (less than the minimal distance between z_0 and the other critical elements of T_f). Let $\bar{z} \in U(\bar{t}, l, z_0) \setminus \{z_0\}$. Then the solution $\tilde{z}(\cdot)$ of the autonomous evolution process with $\tilde{z}(\bar{t}) = \bar{z} \neq z_0$ is also a solution of the semiflow T_f with the same initial condition $\tilde{z}(\bar{t}) = \bar{z}$. Then, by the Morse-Smale property of the semiflow T_f the α-limit set and ω-limit set of the orbit of $\tilde{z}(\cdot)$ are two distinct critical elements of T_f. Therefore at least one of these limit sets is at a distance from z_0 larger than l. This implies that the solution $\tilde{z}(\cdot)$ leaves the neighborhood $U(\cdot, l, z_0)$ definitively either forward or backward in time, i.e., $\xi(\cdot)$ is nonrecurrent by Definition 8.1. Similar arguments show that trajectories corresponding to hyperbolic periodic orbits of T_f are nonrecurrent outside their cylinders (Definition 8.2). This completes the proof. $\qquad \square$

This Theorem establishes the class of Morse-Smale autonomous evolution processes on $\mathcal{B}$. We now consider the evolution processes obtained by nonautonomous perturbations of the elements in this class. From here on we study processes derived from nonautonomous differential equations like the problems analyzed in Section 7 for the persistence of isolated invariant manifolds.

Let $\mathcal{B} \subseteq X$ be Banach spaces. Assume L is a linear operator generating a C_0-semigroup of bounded linear operators in X and let $g \in C^1(\mathcal{B}, X)$. As in Section 7, we consider either the finite dimensional case $\mathcal{B} = X = \mathbb{R}^n$ or, in infinite dimensions, the case of $-L$ sectorial and $\mathcal{B} = X^\alpha$, with appropriate $\alpha \in (0, 1)$. We also assume that the nonlinearity $g \in C^1(\mathcal{B}, X)$ satisfies conditions ensuring the existence of pullback attractors $\{A_0(t), t \in \mathbb{R}\}, \{A_\varepsilon(t), t \in \mathbb{R}\}$ for the evolution processes generated by the differential equations (18), (19), (equations (30), (31) below). Let $\mathcal{F} = C_{\varepsilon_0}(\mathbb{R}, C^1(\mathcal{B}, X))$ be the metric space of parameters, for ε_0 sufficiently small.

Theorem 9.3. *If the autonomous evolution process on $\mathcal{B}$ generated by*

$$\dot{z} = Lz + g(z) \tag{30}$$

is Morse-Smale, then for every $f_\varepsilon \in \mathcal{F}$ the nonautonomous evolution process on $\mathcal{B}$ generated by the perturbed equation

$$\dot{z} = Lz + g(z) + f_\varepsilon(t, z) \tag{31}$$

is also Morse-Smale.

Proof. Let $S_0(\cdot, \cdot)$ and $S_\varepsilon(\cdot, \cdot)$ denote the evolution processes on $\mathcal{B}$ generated by (30) and (31), respectively. As we proved in Theorem 9.2 the set $\mathcal{M}_0^h$ of graphs of hyperbolic solutions of $S_0(\cdot, \cdot)$ is the set of graphs of hyperbolic fixed points z_0 of (30). Hence by Section 7.1, for $0 < \varepsilon \leq \varepsilon_0$, the set $\mathcal{M}_\varepsilon^h$ of graphs of hyperbolic solutions of $S_\varepsilon(\cdot, \cdot)$ is the set of graphs of hyperbolic solutions $z_\varepsilon(\cdot)$ of (31). Moreover, the unstable manifolds $W^u(\alpha_\varepsilon)$ and the local stable manifolds $W_{loc}^s(\alpha_\varepsilon)$ of these solutions $z_\varepsilon(\cdot)$ are C^1 ε-close to the corresponding unperturbed manifolds, $W^u(\alpha_0)$ and $W_{loc}^s(\alpha_0)$, [25, Chapter 9].

Let $\mathcal{M}_0^{nh}$ denote the set of tubes $\mathcal{M}_0^T$ of the autonomous evolution process $S_0(\cdot, \cdot)$ and let $\mathcal{M}_\varepsilon^{nh}$ denote the set of tubes $\mathcal{M}_\varepsilon^T$ of $S_\varepsilon(\cdot, \cdot)$. Then, each tube $\mathcal{M}_0^T$ is a cylinder generated by a periodic solution $\xi_0(\cdot)$ of (30). By Section 7.2, for $0 < \varepsilon \leq \varepsilon_0$, each tube $\mathcal{M}_\varepsilon^T$ is a normally hyperbolic isolated invariant manifold fibered by graphs of partially hyperbolic solutions $\xi_\varepsilon(\cdot, \tau)$ of (31). Remark that the exponential trichotomy for these solutions $\xi_\varepsilon(\cdot, \tau)$ allows for a small exponential behavior on the central projection, see [35], [2, Section 8]. Also the unstable manifolds $W^u(\mathcal{M}_\varepsilon^T)$ and the local stable manifolds $W_{loc}^s(\mathcal{M}_\varepsilon^T)$ are C^1 ε-close to the corresponding unperturbed manifolds, $W^u(\mathcal{M}_0^T)$ and $W_{loc}^s(\mathcal{M}_0^T)$. Therefore conditions (1) and (2) of Definition 9.1 are satisfied.

Before addressing condition (3) we identify, for simplicity, the graphs α_ε in $\mathcal{M}_\varepsilon^h$ with tubes $\mathcal{M}_\varepsilon^S$ with a single fiber. We also identify the corresponding unstable and local stable manifolds, $W^u(\mathcal{M}_\varepsilon^S) \cong W^u(\alpha_\varepsilon)$, $W_{loc}^s(\mathcal{M}_\varepsilon^S) \cong W_{loc}^s(\alpha_\varepsilon)$. We let $\mathcal{M}_\varepsilon^{hs}$ denote the set of these singular tubes.

To prove condition (3) we consider the case of a hyperbolic solution $z_\varepsilon(\cdot)$ of (31) (hence defining a singular tube $\mathcal{M}_\varepsilon^S$ in $\mathcal{M}_\varepsilon^{hs}$). For a given time $\bar{t} \in \mathbb{R}$ let $U(\bar{t}, l, z_\varepsilon(\bar{t}))$ denote the neighborhood of $z_\varepsilon(\bar{t})$ in $A_\varepsilon(\bar{t}) \subset \mathcal{B}$ with $l > 0$ sufficiently small and let $\bar{z} \in U(\bar{t}, l, z_\varepsilon(\bar{t})) \setminus \{z_\varepsilon(\bar{t})\}$. Note that if $U(\bar{t}, l, z_\varepsilon(\bar{t})) \setminus \{z_\varepsilon(\bar{t})\} = \emptyset$ the pullback attractor of $S_\varepsilon(\cdot, \cdot)$ reduces to $\{z_\varepsilon(t), t \in \mathbb{R}\}$ corresponding to the unique hyperbolic solution $z_\varepsilon(\cdot)$, and the nonrecurrence is trivial.

If $\bar{z} \notin W_{loc}^s(\mathcal{M}_\varepsilon^S)$, for some large time $\bar{t} + \overline{T}_1$ the solution $\tilde{z}_\varepsilon(\cdot)$ of (31) with $\tilde{z}_\varepsilon(\bar{t}) = \bar{z}$ is either in some neighborhood of a tube $\mathcal{M}_\varepsilon^T \in \mathcal{M}_\varepsilon^{nh}$, or else in some neighborhood of another singular tube $(\mathcal{M}_\varepsilon^S)' \in \mathcal{M}_\varepsilon^{hs}$ with $(\mathcal{M}_\varepsilon^S)' \neq \mathcal{M}_\varepsilon^S$. If $\tilde{z}_\varepsilon(\bar{t} + \overline{T}_1)$ is in the local stable manifold of $\mathcal{M}_\varepsilon^T$ or $(\mathcal{M}_\varepsilon^S)'$ then the solution $\tilde{z}_\varepsilon(t)$ does not return to the neighborhood $U(t, l, z_\varepsilon(t))$ for $t > \bar{t} + \overline{T}_1$. If, on the other hand, $\tilde{z}_\varepsilon(\bar{t} + \overline{T}_1)$ is not in the local stable manifold of $\mathcal{M}_\varepsilon^T$ or $(\mathcal{M}_\varepsilon^S)'$ we repeat the same argument with a neighborhood of a solution with graph in $\mathcal{M}_\varepsilon^T$ or $(\mathcal{M}_\varepsilon^S)'$, shrinking if necessary the original neighborhood. Then we obtain an increasing sequence of times $\overline{T}_j, j = 1, \ldots, n$, such that $\tilde{z}_\varepsilon(\bar{t} + \overline{T}_j)$ is in the neighborhood of a solution with graph in some tube in $\mathcal{M}_\varepsilon^{hs} \cup \mathcal{M}_\varepsilon^{nh}$. Therefore we obtain a sequence of tubes visited by the solution $\tilde{z}_\varepsilon(\cdot)$. Moreover all the tubes in this sequence are distinct because the invariant manifolds of the tubes of the perturbed evolution process $S_\varepsilon(\cdot, \cdot)$ are ε-close to the invariant manifolds of the tubes of the autonomous evolution process $S_0(\cdot, \cdot)$ which by transversality does not have cycles. We conclude that $\tilde{z}_\varepsilon(t)$ eventually reaches the local stable manifold of a tube distinct from $\mathcal{M}_\varepsilon^S$, hence cannot return to the neighborhood $U(t, l, z_\varepsilon(t))$ for $t > \bar{t} + \overline{T}_n$.

If $\bar{z} \in W_{loc}^s(\mathcal{M}_\varepsilon^S)$, then $\bar{z}$ is in the unstable manifold of some tube $\mathcal{M}_\varepsilon^\alpha \in \mathcal{M}_\varepsilon^{hs} \cup \mathcal{M}_\varepsilon^{nh}$ distinct from $\mathcal{M}_\varepsilon^S$. Indeed this follows from the continuity properties of the pullback attractor (see Propositions 2, 3, and also [4, Theorem 2.21]) since $\tilde{z}_\varepsilon(\cdot)$ is a global bounded solution of $S_\varepsilon(\cdot, \cdot)$, therefore in $\{A_\varepsilon(t), t \in \mathbb{R}\}$. Hence there exists a $\overline{T}_0 > 0$ such that the solution $\tilde{z}_\varepsilon(\cdot)$ is in a neighborhood of the isolated invariant

set $\mathcal{M}_\varepsilon^\alpha$ for $t < \bar{t} - \overline{T}_0$. We conclude that $\tilde{z}_\varepsilon(t)$ leaves the neighborhood $U(t, l, z_\varepsilon(t))$ and cannot return to it for all $t < \bar{t} - \overline{T}_0$. This shows that the trajectory of $\tilde{z}_\varepsilon(\cdot)$ is nonrecurrent by Definition 8.1. Similar arguments show that the trajectory $\xi_\varepsilon(\cdot)$ of a partially hyperbolic solution $z_\varepsilon(\cdot)$ of $S_\varepsilon(\cdot, \cdot)$ in a tube $\mathcal{M}_\varepsilon^T \in \mathcal{M}_\varepsilon^{nh}$ is nonrecurrent outside $\mathcal{M}_\varepsilon^T$.

This concludes the proof of condition (3) of Definition 9.1 and shows that the perturbed evolution process $S_\varepsilon(\cdot, \cdot)$ is Morse-Smale. $\qquad\square$

10. Examples.

10.1. **Morse-Smale evolution process on $\mathcal{B} = \mathbb{R}^2$.** The initial system (1) of ordinary differential equations on $\mathcal{B} = \mathbb{R}^2$ provides an example of a nonautonomous perturbation of an autonomous Morse-Smale system in finite dimensional space. Then, by Theorem 9.3 for small $\delta > 0$ the nonautonomous evolution process $S_\delta(\cdot, \cdot)$ generated by (1) is also Morse-Smale.

10.2. **Morse-Smale evolution process on $\mathcal{B} = H^{2\alpha}(S^1)$.** To provide an example in infinite dimensional space $\mathcal{B}$ we consider the scalar parabolic semilinear differential equation

$$u_t = u_{xx} + g(u, u_x) + \varepsilon f(t, x, u, u_x) \ , \ x \in S^1 = \mathbb{R}/\mathbb{Z} \ , \tag{32}$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ is C^2 and satisfies dissipative conditions of the form $ug(u, 0) < 0$ for all large $|u|$, and $|g(u, p)| \le c(1 + |p|^\gamma)$ with $c > 0, 0 \le \gamma < 2$, uniformly for u in compact intervals, and $f : \mathbb{R} \times S^1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is also C^2 and bounded. To take $x \in S^1$ here is equivalent to consider periodic boundary conditions

$$u(t, 0) = u(t, 1) \ , \ u_x(t, 0) = u_x(t, 1) \ .$$

Under these conditions equation (32) generates an evolution process on the fractional power space $\mathcal{B} = X^\alpha = H^{2\alpha}(S^1), 3/4 < \alpha < 1$. Moreover, by the results in [7] for small values of $\varepsilon > 0$ the evolution process $S_\varepsilon(\cdot, \cdot)$ has a pullback attractor $\{A_\varepsilon(t), t \in \mathbb{R}\}$. The unperturbed differential equation $(32)_{\varepsilon=0}$ has been extensively considered in the literature and is well understood, see [1, 19, 17] and the references therein. In general, the semiflow $T_g(\cdot)$ generated by the autonomous differential equation $(32)_{\varepsilon=0}$ has equilibrium solutions and also periodic orbits. Furthermore all the equilibria and periodic orbits are hyperbolic generically in g, see [28]. Therefore, we assume that g is such that all the equilibria and periodic orbits of $T_g(\cdot)$ are hyperbolic. Then, by the automatic transversality of all the unstable and local stable manifolds of its critical elements, $T_g(\cdot)$ is a Morse-Smale semiflow of transformations of $\mathcal{B}$, (see [17], [19] and [29] for these specific boundary conditions and also the pioneering work [26]). In fact, in the class of problems $(32)_{\varepsilon=0}$ the automatic transversality between stable and unstable manifolds of periodic orbits and other critical elements, [17, 19], also holds for pairs of equilibria due to the nonexistence of heteroclinic connections between homoindexed equilibria, [29, 19]. So the autonomous evolution process $S_0(t, \tau) = T_g(t - \tau)$ is Morse-Smale on $\mathcal{B}$ by Theorem 9.2 and the nonautonomous evolution process $S_\varepsilon(\cdot, \cdot)$, for sufficiently small $\varepsilon > 0$, is also Morse-Smale on $\mathcal{B}$ by Theorem 9.3.

10.3. **Asymptotically autonomous Morse-Smale evolution process.** Next, we consider the case of asymptotically autonomous evolution processes. Let $S(\cdot, \cdot)$ denote an evolution process on $\mathcal{B}$ generated by the differential equation

$$\dot{z} = Lz + g(z) + f(t, z) \ , \tag{33}$$

with L and g satisfying the same conditions of Theorem 9.3, but with nonautonomous term $f \in C_b(\mathbb{R}, C^1(\mathcal{B}, X))$ satisfying

$$\lim_{t \to +\infty} \sup_{z \in \mathcal{B}} \left(\|f(t, z)\|_X + \|f_z(t, z)\|_{\mathcal{L}(\mathcal{B}, X)} \right) = 0 \ .$$

Then $S(\cdot, \cdot)$ is asymptotically autonomous and its limiting behavior should approach the autonomous evolution process $S_\omega(\cdot, \cdot)$ generated by the differential equation

$$\dot{z} = Lz + g(z) \ . \tag{34}$$

If we replace the time dependence in (33) by a smooth monotone function $\mathbb{R} \ni t \mapsto h_\eta(t) \in \mathbb{R}$ satisfying

$$h_\eta(t) = \begin{cases} \eta, & \text{if } t \le \eta \\ t, & \text{if } t > \eta + 1 \end{cases}$$

we obtain an asymptotically autonomous evolution process which is autonomous in the past (for $t \le \eta$). Let $S_\eta(\cdot, \cdot)$ denote the evolution process generated by the differential equation

$$\dot{z} = Lz + g(z) + f(h_\eta(t), z) \ . \tag{35}$$

Then $f \in C_\varepsilon(\mathbb{R}, C^1(\mathcal{B}, X))$ with $\varepsilon = \varepsilon(\eta) \to 0$ as $\eta \to +\infty$. Hence, if the autonomous process $S_\omega(\cdot, \cdot)$ is Morse-Smale on $\mathcal{B}$, then for large values of η the asymptotically autonomous evolution process $S_\eta(\cdot, \cdot)$ is also Morse-Smale on $\mathcal{B}$ by Theorem 9.3. This holds even if $S(\cdot, \cdot)$ is not Morse-Smale, exhibiting different behaviors in the past and in the future.

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E-mail address: rczaja@math.us.edu.pl
E-mail address: wamoliva@math.ist.utl.pt
E-mail address: crocha@math.ist.utl.pt